DOCUMENT RESUME

ED 184 836 SE 030 452

AUTHOR	Steffe, Leslie P., Ed.; And Others
TITLE	Soviet Studies in the Psychology of Learning and Teaching Mathematics, Volume VIII: Methods of Teaching Mathematics.
INSTITUTION	Chicago Univ., Ill.; Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	75
GRANT	NSF-G-8847(MA)
NOTE	290p.; For related documents, see SE 030 451-458, ED 042 528, ED 042 632-633, and ED 080 382-384
EDRS PRICE	MF01/PC12 Plus Postage.
DESCRIPTORS	Elementary Secondary Education; Independent Study; Mathematical Applications; *Mathematics Education; *Mathematics Instruction
IDENTIFIERS	School Mathematics Study Group; *Soviet Research; *USSR

ABSTRACT

This is one of a series that is a collection of translations from the extensive Soviet literature of the past 25 years on research in the psychology of mathematics instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English. The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. This series should assist in opening up avenues of investigation to those who are interested in broadening the foundations of their profession. This volume contains four articles: Principles, Forms, and Methods of Mathematics Instruction; The Relation Between Mathematics Instruction and Life; The Pupil's Activity as a Necessary Condition for Improving the Quality of Instruction; and Independent Work for Pupils in Arithmetic Lessons in the Early Grades. (MK)

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SOVIET STUDIES IN THE PSYCHOLOGY OF LEARNING AND TEACHING MATHEMATICS

VOLUME VIII

ED184836

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EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

[illegible]

Mary L. Charles
NSF

SCHOOL MATHEMATICS STUDY GROUP
STANFORD UNIVERSITY
AND
SURVEY OF RECENT EAST EUROPEAN
MATHEMATICAL LITERATURE
THE UNIVERSITY OF CHICAGO

SE 030 452

SOVIET STUDIES
IN THE
PSYCHOLOGY OF LEARNING
AND TEACHING MATHEMATICS

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VOLUME VIII

METHODS OF TEACHING MATHEMATICS

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Financial support for the School Mathematics Study Group and for the Survey of Recent East European Mathematical Literature has been provided by the National Science Foundation.

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PREFACE

The series Soviet Studies in the Psychology of Learning and Teaching Mathematics is a collection of translations from the extensive Soviet literature of the past twenty-five years on research in the psychology of mathematical instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. The series is the result of a joint effort by the School Mathematics Study Group at Stanford University, the Department of Mathematics Education at the University of Georgia, and the Survey of Recent East European Mathematical Literature at the University of Chicago. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English.

Research achievements in psychology in the United States are outstanding indeed. Educational psychology, however, occupies only a small fraction of the field, and until recently little attention has been given to research in the psychology of learning and teaching particular school subjects.

The situation has been quite different in the Soviet Union. In view of the reigning social and political doctrines, several branches of psychology that are highly developed in the U.S. have scarcely been investigated in the Soviet Union. On the other hand, because of the Soviet emphasis on education and its function in the state, research in educational psychology has been given considerable moral and financial support. Consequently, it has attracted many creative and talented scholars whose contributions have been remarkable.*

Even prior to World War II, the Russians had made great strides in educational psychology. The creation in 1943 of the Academy of Pedagogical Sciences helped to intensify the research efforts and programs in this field. Since then the Academy has become the chief educational research and development center for the Soviet Union. One of the main aims of the Academy is to conduct research and to train research scholars in general and specialized education, in educational psychology, and in methods of teaching various school subjects.

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*A study indicates that 37.5% of all materials in Soviet psychology published in one year was devoted to education and child psychology. See Contemporary Soviet Psychology by Josef Brozek (Chapter 7 of Present-Day Russian Psychology, Pergamon Press, 1966).

The Academy of Pedagogical Sciences of the USSR comprises ten research institutes in Moscow and Leningrad. Many of the studies reported in this series were conducted at the Academy's Institute of General and Polytechnical Education, Institute of Psychology, and Institute of Defectology, the last of which is concerned with the special psychology and educational techniques for handicapped children.

The Academy of Pedagogical Sciences has 31 members and 64 associate members, chosen from among distinguished Soviet scholars, scientists, and educators. Its permanent staff includes more than 650 research associates, who receive advice and cooperation from an additional 1,000 scholars and teachers. The research institutes of the Academy have available 100 "base" or laboratory schools and many other schools in which experiments are conducted. Developments in foreign countries are closely followed by the Bureau for the Study of Foreign Educational Experience and Information.

The Academy has its own publishing house, which issues hundreds of books each year and publishes the collections _Izvestiya Akademii Pedagogicheskikh Nauk RSFSR_ [Proceedings of the Academy of Pedagogical Sciences of the RSFSR], the monthly _Sovetskaya Pedagogika_ [Soviet Pedagogy], and the bimonthly _Voprosy Psikhologii_ [Questions of Psychology]. Since 1963, the Academy has been issuing collections entitled _Novye Issledovaniya v Pedagogicheskikh Naukakh_ [New Research in the Pedagogical Sciences] in order to disseminate information on current research.

A major difference between the Soviet and American conception of educational research is that Russian psychologists often use qualitative rather than quantitative methods of research in instructional psychology in accordance with the prevailing European tradition. American readers may thus find that some of the earlier Russian papers do not comply exactly to U.S. standards of design, analysis, and reporting. By using qualitative methods and by working with small groups, however, the Soviets have been able to penetrate into the child's thoughts and to analyze his mental processes. To this end they have also designed classroom tasks and settings for research and have emphasized long-term, genetic studies of learning.

Russian psychologists have concerned themselves with the dynamics of mental activity and with the aim of arriving at the principles of the learning process itself. They have investigated such areas as: the development of mental operations; the nature and development of thought; the formation of mathematical concepts and the related questions of generalization, abstraction, and concretization; the mental operations of analysis and synthesis; the development of spatial perception; the relation between memory and thought; the development of logical reasoning; the nature of mathematical skills; and the structure and special features of mathematical abilities.

In new approaches to educational research, some Russian psychologists have developed cybernetic and statistical models and techniques, and have made use of algorithms, mathematical logic and information sciences. Much attention has also been given to programmed instruction and to an examination of its psychological problems and its application for greater individualization in learning.

The interrelationship between instruction and child development is a source of sharp disagreement between the Geneva School of psychologists, led by Piaget, and the Soviet psychologists.* The Swiss psychologists ascribe limited significance to the role of instruction in the development of a child. According to them, instruction is subordinate to the specific stages in the development of the child's thinking--stages manifested at certain age levels and relatively independent of the conditions of instruction.

As representatives of the materialistic-evolutionist theory of the mind, Soviet psychologists ascribe a leading role to instruction. They assert that instruction broadens the potential of development, may accelerate it, and may exercise influence not only upon the sequence of the stages of development of the child's thought but even upon the very character of the stages. The Russians study development in the changing conditions of instruction, and by varying these conditions, they demonstrate how the nature of the child's development changes in the process. As a result, they are also investigating tests of giftedness and are using elaborate dynamic, rather than static, indices.

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*See The Problem of Instruction and Development at the 18th International Congress of Psychology by N. A. Menchinskaya and G. G. Saburova, Sovetskaya Pedagogika, 1967, No. 1. (English translation in Soviet Education, July 1967, Vol. 9, No. 9.)

Psychological research has had a considerable effect on the recent Soviet literature on methods of teaching mathematics. Experiments have shown the student's mathematical potential to be greater than had been previously assumed. Consequently, Russian psychologists have advocated the necessity of various changes in the content and methods of mathematical instruction and have participated in designing the new Soviet mathematics curriculum which has been introduced during the 1967-68 academic year.

The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. This series should assist in opening up avenues of investigation to those who are interested in broadening the foundations of their profession, for it is generally recognized that experiment and research are indispensable for improving content and methods of school mathematics.

We hope that the volumes in this series will be used for study, discussion, and critical analysis in courses in seminars in teacher-training programs or in institutes for in-service teachers at various levels.

At present, materials have been prepared for fifteen volumes. Each book contains one or more articles under a general heading such as The Learning of Mathematical Concepts, The Structure of Mathematical Abilities and Problem Solving in Geometry. The introduction to each volume is intended to provide some background and guidance to its content.

Volumes I to VI were prepared jointly by the School Mathematics Study Group and the Survey of Recent East European Mathematical Literature, both conducted under grants from the National Science Foundation. When the activities of the School Mathematics Study Group ended in August, 1972, the Department of Mathematics Education at the University of Georgia undertook to assist in the editing of the remaining volumes. We express our appreciation to the Foundation and to the many people and organizations who contributed to the establishment and continuation of the series.

Jeremy Kilpatrick
Izaak Wirszup
Edward G. Begle
James W. Wilson

EDITORIAL NOTES

1. Bracketed numerals in the text refer to the numbered references at the end of each paper. Where there are two figures, e.g. [5:123], the second is a page reference. All references are to Russian editions, although titles have been translated and authors' names transliterated.

2. The transliteration scheme used is that of the Library of Congress, with diacritical marks omitted, except that Ю and Я are rendered as "yu" and "ya" instead of "iu" and "ia."

3. Numbered footnotes are those in the original paper, starred footnotes are used for editors' or translator's comments.

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INTRODUCTION

Leslie P. Steffe

Gibsh, in the paper "Principles, Forms, and Methods of Mathematics Instruction," begins with a statement describing goals of Soviet mathematics teaching in 1958. While some of the goals reflect the influence of dialectical materialism and Communist party directives, others are universal. Of particular interest are the goals emphasizing the social utility of mathematical knowledge, the ability to solve theoretical and practical problems, logical thinking, functional relationships, and spatial representation. These universal goals are of particular significance for school mathematics in the United States today and represent forces that historically have influenced school mathematics in this country. Social utility theory was a strong force in school mathematics circa 1920-40 and is taking on new significance during the 1970's in the face of a resurgence of emphasis on school mathematics programs for "everyman." The ability to solve theoretical and practical problems continues to be of central concern for school mathematics programs and is being highlighted during the 1970's because of forces contributing to the creation of interdisciplinary programs for school children. The value of mathematical studies to logical thinking was emphasized during the most influential era of faculty psychology and is currently receiving attention (albeit different) with a reemphasis on child-centered school mathematics programs and accompanying psychological bases for mathematical learning and reasoning. The goal concerning functional relationships and spatial imagination is also timely in the United States because of the emphasis on transformational geometry and functions in school mathematics and the utilization of functions in the descriptions of children's thought.

The remainder of Gibsh's paper deals with principles, forms, and methods of mathematics instruction. Ten principles of mathematical instruction are stated. The first concerns pupil activity in the classroom. Gibsh characterizes active mental work as "finding independently the method of solving a problem and substantiating the way to solve it (p. 5)." Thus, Gibsh characterizes knowledge acquired through active mental work so that it becomes synonymous with knowledge acquired through discovery or invention--the solution of a problem is necessary, according to Gibsh, for pupils to become active.

Gibsh's next three principles of mathematics instruction are interrelated. First, the basic criterion of success for a pupil's activity in mathematics is the degree of consciousness the pupil shows in mastering the theory. Next, mathematical theory can be presented only in a strict logical sequence. Third, the teacher should imagine precisely what the aim of a lesson is--what new concepts it contains and what new abilities and skills the pupils should obtain from the lesson. Gibsh presents excellent examples, in mathematical contexts, of the meaning of conscious, thorough mastery of study material.

Indeed, in mathematics education, the student is generally expected to become conscious of the mathematics he is learning. This expectation, however, should be duly tempered in its application, especially during the elementary school years. In the elementary school, hardly ever should it be expected, for example, that a child would acquire the distributivity principle as a form. Rather, such principles should be viewed as laws which the child's reasoning obeys in much the same way as transitivity is viewed in cognitive development theory. The principle should be held as inseparable from the content being studied and should not be learned as an abstract form. The spirit of the instruction would be quite different depending on which goal one had in mind--conscious mastery of the material, or not.

Even at the secondary school level, strict adherence to the principle of conscious mastery of material can lead to unnecessary complications. It is generally assumed that mathematical concepts go through different "learning stages" in the sense of stage theory in cognitive development [1]. As not all these "learning stages" involve conscious mastery of the material, modification should be made of the principle to accommodate this newer information.

The problem of sequence in mathematical curriculum is yet to be satisfactorily resolved. Gibsh takes the point of view that "mathematics as a subject of study ... cannot be presented without retaining ... complete logical sequence (p. 11)." His argument is predicated on the basis that mathematics as a science is a well-constructed system of scientific facts and any departure from a strict logical organization in its presentation would entail a deeply rooted distortion of the subject. The issue of sequencing mathematical topics can be dichotomized using the product-process distinction. Gibsh's point of view is most consistent with those who emphasize the products of knowing at the expense of the process of knowing--the "Gagné position." But it is a subtle matter to combine logical sequence of a subject with student performance [3]. One cannot refute Gibsh's position with empirical data only, however, because it entails an epistemology quite different than that held by educators who emphasize the processes of knowing rather than the products of knowing.

Gibsh's fourth principle is consistent with two positive contributions made by the behavioral objectives movement in mathematics education in the United States. These two contributions are that (1) teachers have been compelled to distinguish clearly between performance objectives and instructional objectives for their students, and (2) teachers are required to analyze their daily lessons carefully.

The remaining five principles of mathematics instruction stated by Gibsh concern observance of the scientific principle, accessibility of material, use of visual aids, connection between theory and practice, and character development. Gibsh believes that the level of teaching mathematics should correspond to the mathematics as a science--definitions, axioms, and theorems should be given with impeccable precision, and proofs should be conducted as rigorously as possible. In the observance of the scientific principle, Gibsh states that "a proof that is not rigorous, insufficiently substantiated, gives the pupils only surface knowledge .... Facts that cannot yet be proved ... should be accepted without proof (p. 19)." Such a rigorous adherence to the scientific principle precludes justification (or "proofs") not based on an axiomatic method. Certainly, one need not insist on axiomatic method in school mathematics to create convincing arguments for pupils [4].

Gibsh takes the position that mathematics can be made more accessible through the use of visual aids in instruction. Instruction is visual "if it is based on the pupils' direct perception of objects ... by their sense organs (p. 29)." As an example of visual instruction, Gibsh believes that a "child finally grasps the concept of the number "five" only after repeated observations of groups of homogeneous objects appearing as concrete representatives of this number "five," images accessible to direct perception (p. 29)." Formation of a stock of static images of homogeneous groups of five certainly aids a child in "knowing" five, but does not account for the mental operations performed on those images--operations which enable the child to think of five simultaneously as five and, for example, one and four. It is well known that children are able to recognize collections of objects up to (and often) including five but, yet, display an inability to solve the class inclusion problem--a logical requirement for a conception of whole number.

Gibsh's insistence on a connection between theory and practice is consistent with the interdisciplinary approach to school mathematics and science programs and the creation of school mathematics programs for "everyman." Those mathematics educators interested in these issues will certainly find reinforcement in this principle.

Three forms of mathematics instruction and three methods of mathematics instruction are also discussed. The forms are the heuristic form, the laboratory form, and the lecture form. The heuristic form of instruction "consists of a sequential system of expediently composed and distributed questions to which pupils given answers within their capacity ...(p. 40)." This description is reminiscent of a guided discovery technique of instruction as pointed out by Gagné[2] in the learning of principles. Whether one would encounter the heuristic form of instruction in its pure form in actual practice is not certain because it requires the pupils to answer successfully the questions posed--a highly unlikely event. It is, however, an ideal form of instruction toward which a teacher can strive, given knowledge of it.

The laboratory form of instruction, as outlined by Gibsh includes the solution of examples and problems during class, execution of graphic exercises, execution of measuring tasks, and the execution of model-making tasks. The laboratory form is based on the use of visual aids. While no mention is made of psychological theory usually taken as a base for the laboratory form of instruction, the points made in its use are cogent--especially the use of model-making tasks. These tasks are reminiscent of Dienes' [1] stage of representation in mathematical learning.

The methods of mathematics instruction are distinguished from the forms of mathematics instruction in that the former adheres to methods of scientific investigation. The methods described are the Analytic-Synthetic, the Inductive, and the Deductive. The Analytic-Synthetic method resembles Polya's heuristics and can be utilized profitably, especially in mathematical instruction in the secondary school. The inductive method is separated into two parts, incomplete induction and complete induction. The difference resides in the number of observations made. The deductive method consists in applying deductive reasoning to establish the truth of propositions.

Maslova and Semushin, in the paper "The Relation Between Mathematics Instruction and Life," concentrate on one of the ten principles of mathematics instruction for "everyman." While this paper falls short in specification of the content and methods necessary in school mathematics to enable an educated citizen to use mathematics in his everyday life, useful suggestions are made. Throughout mathematics instruction, the teacher must show how mathematics reflects the real world; develop pupils' abilities and skills essential in everyday life and in socially useful, productive work; establish a close connection between methods used in problem solving in school and those in industry; and develop pupils' ability to give mathematical form to practical problems.

Gibsh, in the paper " The Pupil's Activity as a Necessary Condition for Improving the Quality of Instruction,"elaborates on topics presented in his previous paper in this volume. Of particular interest is his assertion that, whenever possible, each topic of the school mathematics course should begin with a statement of the fundamental question to be answered therein. This statement of the fundamental question is reminiscent of Ausubel's advance organizers, even though it does not have all of the same characteristics. As such, it is an interesting classroom procedure whose utility could be ascertained through experimentation. Other topics receiving elaboration are: the heuristic *method of instruction, the teacher's role* in developing the pupil's ability to find methods of solving problems independently, independent execution of exercises by the pupils, and laboratory work for the pupils.

Moro, in his paper "Independent Work for Pupils in Arithmetic Lessons in the Elementary Grades," offers a classification of pupils' independent work based on classroom observations and a juxtaposition of various theoretical points of view. The first class of independent work identified is based on the criterion of the pedagogical aim of the independent tasks. Here, two basic groups of tasks were identified--instructive tasks and checking tasks. The second class of independent work identified is based on the criterion of the nature of the activity demanded of the pupils: Imitative activities, independent application of knowledge and skills acquired earlier with teacher guidance but where the object of application is distinct from that worked on in the acquisition of the knowledge and skills to be applied, and creative tasks where the children must pose the question and seek ways of solution. The third class of independent work is based on the criterion of the curriculum material on which work is done--e.g., concept formation tasks, problem-solving tasks, or practical work.

Moro, then, views independent work more broadly than does Gibsh. In fact, Gibsh's conception of independent work can be categorized under the second class of independent work identified by Moro. Moro's distinction between work that is or is not independent is based mainly on whether the work is done without teacher guidance or with teacher guidance, respectively.

Teachers of elementary school mathematics as well as methodologists will find Moro's manuscript particularly valuable because of the rich and detailed examples of ways of conducting pupils' independent work. Long expositions, with many examples, are given concerning teaching new material through independent work. Topics covered include assignments for independent work including instructing children in problem solving, independent tasks of a practical nature, a system for conducting children's independent work, and independent work for the first two grades by topic. While the material is somewhat dated, it is still relevant to the critical problem of individualizing instruction faced by the elementary schools. Too often, computerized managerial systems of individualizing instruction do not show operational evidence of a thoughtful analysis of children's independent work and of a balance between activities directed or not directed by the teacher. For the large share of schools not operating on a computer managed system of instruction, the analysis of independent work given by Moro is timely. Certainly, it ought to stimulate teachers and supervisors of mathematics to conduct a careful analysis of their daily teaching activities, creating an appropriate balance between independent and nonindependent work--and between the various types of independent work--expected of students.

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PRINCIPLES, FORMS, AND METHODS OF MATHEMATICS INSTRUCTION*

I. A. Gibsh

The Subject and Methodological Problems of Teaching Mathematics

The Party directives concerning the school, which determine the content and directions of all Soviet education, have formed the basis for the teaching of the individual disciplines, including mathematics. At present, mathematics teaching in the Soviet general education school has the following tasks:

1. To communicate to the pupils the knowledge, abilities and skills that constitute the bases of mathematical science and have the most educational and practical value.

2. To develop in the pupils a dialectical materialist world-view during the instruction process.

3. To develop each pupil's ability to present mathematically a problem concerning the quantitative and spatial relationships of the real world and to apply the acquired knowledge and skills for an independent solution of the problem.

4. To clarify, as thoroughly as possible, the content of the school mathematics course, primarily the idea of functional relationship, and to develop the pupils' spatial conceptions and a lively spatial imagination, which they must have for study of geometry and of the many related subjects, as well as for the development of their practical activity.

5. To make the pupils' acquired mathematical knowledge and skills an instrument for solving the problems of practical living.

6. To establish skills in logical thinking and properly (logically and grammatically) constructed speech.

7. To inculcate, in every way possible, in mathematics lessons: (a) Soviet patriotism and national pride; (b) enthusiasm for science; (c) the ability to persevere in overcoming difficulties; (d) operating systematically, the habit of self-checking, attention, and accuracy in doing any kind of assignments.

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*From Proceedings [Izvestiya] of the Academy of Pedagogical Sciences of the RSFSR, 1958, Vol. 92 , pp. 95-148. Translated by David A. Henderson and Joan W. Teller.

Let us examine each of these.

1. Above all, Soviet education, as distinct from pre-revolutionary education, has isolated three aims which should be pursued by school instruction -- the pupil should acquire not only knowledge in the subject, but abilities and habits as well. That is, his theoretical knowledge should be inseparable from practice, supplementing and reinforcing it. This basic requirement is dictated by methodological and educational considerations and corresponds directly to the essence of polytechnic instruction.

2. The only true science is that which realistically describes and illuminates our environment; a science does this only if it is based on a materialist world-view. A proper explanation of natural phenomena cannot help being based on a materialistic perception of them; therefore, the proper teaching of the natural sciences, among which is mathematics, is organically connected with the process of developing a purely scientific, materialist world-view, which is the basis for all training that the pupils acquire in school and which will guide them in their further theoretical and practical activity.

3. Arithmetic, algebra, and trigonometry provide a powerful means for solving and investigating a great number of theoretical and especially practical problems. Many theoretical and practical problems may be solved and investigated by composing and solving equations and inequalities and systems of equations and inequalities, which express analytically the conditions of the corresponding problem. Mastery of the method of equations, as we may call this means of resolving and investigating mathematical problems, is one of the most important indices of the level and depth of the pupil's mathematical knowledge and, to an even greater degree, of his ability to apply this knowledge in practice, which constitutes the most essential element of polytechnic education.

Geometry also affords the pupil active means of solving a great number of problems connected with man's practical activity. We must acknowledge that an even higher mathematical development, an ability to use initiative, and a capacity for elementary creativity are required in this field.

4. The Soviet methodology of teaching mathematics requires that this instruction be profoundly interesting, that it reveal fully and comprehensively each concept in the mathematics course. Only such instruction can

fully effect a *conscious and thorough mastery* of the material, and this is one of the most important criteria of the pupil's mathematical *enlightenment*, in our Soviet interpretation of the word.

Of the basic ideas of the school mathematics course, first place should be given to presenting the idea of *functional relationship*, which is the best expression of the essence of mathematics as a science concerned with the dialectic aspect of the quantitative and spatial relationships of the real world.

Using the method of equations and the idea of functional relationship makes it possible to study the quantitative relationships of reality in their static and dynamic states. And, in these forms, the spatial relationships of reality should be studied, as the teacher works on developing proper spatial conceptions and a lively spatial imagination in the pupils. Mastery of spatial conceptions and the presence of a spatial imagination are another criterion of the pupil's mathematical *enlightenment*.

5. It has already been shown, in examining the first requirement above, that the mathematics education of the middle-school graduate should consist of the knowledge, abilities and skills he has acquired. This requirement must be considered in all of its implications.

Confronting the methodology of teaching mathematics is the problem of finding ways to conduct practical mathematical activities which would most closely approach life, the real environment that awaits the school graduate. In particular, this work can be more successful if the pupil, after having found a theoretical solution to a problem, is able, as they say, to reduce it to a number, to present it finally in numerical form with the proper degree of accuracy, and, if needed, in a visual form, using diagrams or graphs. For acquiring this knowledge, the pupil must have skill in using computing instruments (the arithmometer, the slide rule) and various tables, as well as in obtaining appropriate information from approximate calculation. It goes without saying that a person who is solving practical problems should have a firm knowledge of mathematics, at a sufficient level of development.

6. The distinguishing and determining feature of the Soviet school is that its guiding principle is the training of people who have the best traits of Soviet man--the builder of a communist society. This man should

be distinguished by his initiative, his ability for a creative solution of problems, and his habits of logical thinking. Properly and deliberately organized mathematics instruction gives the teacher the greatest opportunities for training persons of this type, and although teachers in all disciplines are working toward this goal, the best, most advantageous position is held by the teacher of mathematics—a subject in which initiative and creative abilities and the application of logical thinking are often crucial.

7. The same conception of each member of our Soviet society as a man who participates actively in all the diverse and multifaceted operations of a nation obliges the school to train in his progeny these moral qualities that are needed in life, listed above.

This survey of problems, which face mathematics teaching in the Soviet school, gives a certain notion of the scope and nature of the problems constituting the subject matter and content of the *methodology of teaching mathematics.*

The methodology of teaching mathematics, stemming from the general theory established by the Soviet school, establishes what principles should underlie mathematics instruction in the Soviet school, what forms this instruction should take, and by what methods it should be realized.

The essence and content of the methodology of teaching mathematics as a science is wholly determined by the essence and content of the problems facing the school in mathematics instruction.

Using the definition established by Soviet education, one may characterize the Soviet methodology of mathematics teaching having these points as its subject of study: (1) a communistic upbringing of the pupils during their instruction in the school course in mathematics; (2) instruction in the school mathematics course, i.e., the communication of knowledge, abilities, and skills in mathematics to the pupils; (3) training of the pupils in the field of mathematics, i.e., attaining, with the aid of instruction, that level of development of the pupils in mathematics which would afford the middle-school graduate an opportunity to be successful in his chosen vocation.

Principles of Mathematics Instruction

Pupil Activity in Class

One of the most essential requirements for facilitating the improvement of the quality of instruction is the awakening of the pupils' interest and activity. The state of activity, as contrasted with passivity, occurs when the pupils exert a certain mental tension, a certain effort of thought, directed at solving a problem assigned to them. The pupil's will, which forces him to think over the proposed questions, takes part in this operation. The more interesting the problem, the more satisfaction the pupils who perform such thinking have, and the more successfully they move along in solving it. It is obvious that knowledge acquired by active mental work (finding independently the method of solving a problem and substantiating the way to solve it) is the soundest. Facts thus established can always be reproduced, and the methods of argument and the conclusions arrived at by the pupil in solving one problem can be applied in solving other problems.

The teacher has several means of making the class participate, the major ones of which are discussed below.

1. The exposition of each chapter and, where possible, each topic, of the mathematics course should begin with a statement of the question. This statement serves as a short introduction, a preamble to the chapter. It establishes a connection with previous material and explains the basic aim of the topic or problem to be solved. This introduction gives a perspective to the listeners, it awakens an interest in solving the problem and often outlines general ways of solving it. Here are some examples:

a. The introduction of fractional and negative numbers makes it necessary to increase the stock of numbers, since the "old" numbers are insufficent for expressing the result of measuring quantities not containing the unit of measurement an integral number of times and the result of measuring directed quantities. This same idea must guide the establishment of the concept of irrational and imaginary numbers.

b. The concept of direct and inverse proportion can be established by isolating the essential properties ("it is increased or decreased by so many times") that distinguish this type of functional relationship from other types.

c. The triangle can be distinguished from all other polygons by its rigidity; the geometric expression of this property is the fact that from the corresponding equality of the sides of two triangles, one can derive the corresponding equality of the angles. The question of the possibility of the converse leads to the idea of similarity.

d. The pupils can be familiarized with the ideas of symmetry, proportion, and similarity by beginning with the examination of objects in nature (leaves of trees, flower petals, etc.) and of the most nearly perfect productions of architecture, painting, and sculpture.

e. Properties of the circle may be isolated by comparing it with a straight line, which has both similar and sharply distinguishing characteristics. Even familiarizing the pupils superficially with other plane curves (the ellipse, parabola, hyperbola, sine curve, spiral) and, perhaps, spatial ones (for instance, the helix) produces animation and variety.

f. In approaching the concept of the second-degree equation (and equations of higher degrees in general), one may direct the pupils' attention to the fact that the simplest equation, whose roots may be indicated by the number α and β, has the form $(x - \alpha)(x - \beta) = 0$, i.e. the form $x^2 - (\alpha + \beta) x + \alpha\beta = 0$. The question arises whether it is possible to maintain that, conversely, every equation of the type $ax^2 + bx + c = 0$ has two roots.

g. The unit "Metric Relations in the Triangle" contains the presentation and solution of the problem of analytically (as distinct from graphically) finding some elements of a triangle, given its other elements. A number of facts which manifest the existence of a functional relationship among elements of a triangle (features of the congruence of triangles) may be replaced by facts expressing this relationship analytically.

h. General properties of the exponential function should be a generalization of the properties of the exponent with a positive base and exponents which are positive integers already established in arithmetic: "In raising a proper or an improper fraction to a power with an exponent which is a positive integer, it remains respectively proper or improper"; "If the base is a fraction that is proper (improper, distinct from one), with an increase of the exponent the fraction is decreased (increased)."

i. The unit on the circumference and the area of a circle (in the ninth grade) may be begun with some brief information on Archimedes and his work in which he uses the "method of exhaustion."

j. The introduction of the complex numbers in the form of a pair of real numbers can be given naturally to the pupils in passing from the number axis, each point of which is in correspondence with a definite real number, to the plane, on which a given system of rectangular coordinate axes has a definite pair of real numbers corresponding to each of its points.

k. The transition from the geometric arc and the geometric angle to the directed arc and the directed angle in the trigonometry course may be made analogously with the transition from the geometric segment to the vector placed on the axis.

ℓ. The unit on "Similarity of Figures" can be begun by more accurately defining similar figures having identical shape but different dimensions, which were originally described in the sixth and seventh grades. Later, with the aid of construction (by hand), the pupils become convinced that the corresponding equality of the angles of two polygons generally (where $n > 3$) does not involve the proportionality of the sides. Finally, they can examine figures that are exceptional in this respect--triangles--and establish conditions of their similarity, from which two theorems, formulated as direct and inverse theorems, express the above relationship between the equality of corresponding angles and the proportionality of the sides of triangles.

2. The greatest success in animating the class may be obtained by conducting the instruction according to the analytic-synthetic method, which demands thorough preliminary work and craftsmanship of the teacher. The detailed outline of the individual topics of the course, assuming it is done according to the analytic-synthetic method, is a difficult but quite interesting problem which must be solved by the teacher in his daily creative work. Each such outline should contain a whole system of problems which the teacher has to ask the pupils; the solution of each problem should be complete and comprehensive, and should involve the class to the utmost. Examples of outlines such as these are given in the section on the analytic-synthetic method, pages 54-67.

3. A state of necessary activity facilitates the pupils' independent completion of the mathematics assignments in class and at home. The teacher should therefore see that the pupil is forced to seek out, as often as possible, the solution of theoretical and practical questions (examples, problems) independently, applying all his efforts and doing determined mental work, without leaning on the "urgings" of the teacher, who mistakenly considers them a useful "help" for the pupil. This does not mean, finally, that the pupil should be deprived of the teacher's general guidance in his independent solution of the problems assigned to him.

The Pupil's Conscious and Firm Mastery of the Study Material

The basic criterion for the success of the pupil's activities in mathematics is the degree of consciousness he shows in mastering theory and in doing exercises. What is meant by a conscious, thorough mastery of the study material? Let us examine some examples.

Example 1. The unit on parallelograms is begun with a definition of the term "parallelogram," i.e., the term "parallelogram" replaces, according to agreement, the words "a convex quadrangle whose opposing sides are mutually parallel (property α)." This property is established in the presence of a convex quadrangle by agreement. It is then established that if the quadrangle has property α, it also has other properties β, γ, and δ, relating to its sides, angles, and diagonals. A question then arises: "Is the converse applicable—do each of the properties β, γ, δ entail property α?" It turns out that they do. Each of the properties β, γ, δ, therefore, can be taken as equivalent to α as the property defining the parallelogram. The properties β, γ, δ are then conditions by which one may conclude whether a convex quadrangle is a parallelogram. The precise understanding of this role of properties β, γ, δ and their relation to property α would wholly determine conscious mastery of the topic. These ideas relate not only to the unit on parallelograms, of course, but to many others.

Example 2. Having thoroughly studied the unit on measurement of segments (magnitudes), the pupil should answer the following questions completely and with substantiation: (a) Do segments A = 4.7 cm. and B = 8.3 cm. have a common measure? What is it? (b) Segment A is equal to

$\frac{3}{5}$ of segment B. Find the greatest common measure of these segments, (c) Segment A is equal to $\frac{15}{8}$ of segment B. What number will be used to express the result of a decimal measurement of segment A by segment B? (d) The same question in relation to segments A and B, linked by the equation $A = \frac{17}{15}B$ or $A = \frac{5}{3}B$. (e) Under what circumstances will the result of measuring a segment be expressed by a rational number? By an irrational number? Why?

Each topic that the class undertakes should be thoroughly thought out by the teacher, and analyzed by him with regard to various aspects--what its aims and basic content are; what the connection between this topic and the previous material is; how the ideas of previous material can be used to present the current topic as a natural development of ideas that are already familiar to the pupils; how to isolate from the topic being studied all its essential elements which wholly define its content.

Example 3. When beginning the exposition of the unit on second-degree equations, the teacher should review with the pupils all information concerning the basic concepts of first-degree equations and, using them skillfully, should make the pupils understand new concepts and facts concerning the unit on second-degree equations. The teacher should explain that the concept of the equation and its root and the plan for solving an equation remain unchanged, but that the second-degree equation differs from the first-degree equation by the number of roots and the means of solution. If the teacher begins the exposition of this topic by asking the pupils to write an equation which would have the numbers 3 and 5 as its roots, and then leads them to the idea that this requirement would be satisfied by the equation $(x - 3)(x - 5) = 0$, which could be transformed to $x^2 - 8x + 15 = 0$, then it would be natural to ask whether any equation of the type $x^2 + px + q = 0$ has two roots, and how they can be found. It should seem completely reasonable to the pupils that this question is equivalent to the question whether any trinomial of the type $x^2 + px + q$ can be represented as the product of two factors that are linearly related to the unknown x. Further arguments establish that such factoring can be done only if $p^2 - 4q \geq 0$. This method of presenting the topic completely develops the basic idea that the roots of an integral rational second-degree function can be found if this function is decomposible into linear factors.

If the pupil grasps this basic idea fully, he has mastered the topic with sufficient awareness and will be able, with that awareness, to go on to solve analogous questions in his future solution of equations of higher degrees.

Example 4. From geometry we know that a triangle is defined by three of its independent elements, of which at least one is linear. Therefore, beginning the unit "Metric Relations in the Triangle" (mentioned in example g in the section "Pupil Activity in Class"), the teacher can bring to the pupils' attention the fact that to determine the length of any side of a right triangle, it is enough to know the length of its other two sides, with which the length of the unknown side can be connected in one equation. This lesson and the next are in the chapter which takes up the problem of finding this equation, this relation. The relation expressing the Pythagorean theorem is this kind of equation. In the scalene triangle, if we proceed from geometric considerations, it is not enough to know two sides, and, as can be seen from the very method of finding the square of the third side, another element must be given—the projection of one of the given sides onto the other. This projection is subsequently replaced by the product of the side by the cosine of the angle enclosed by the two sides.

Thus, the ability to obtain some metric relation and even a whole system of these relations cannot be considered as sufficient conscious mastery of a topic. It is necessary that the pupil recognize the whole problem—that he begin from some substantiated foresight and think purposefully. Even before inferring the Pythagorean theorem he should be aware that knowing the lengths of two legs of a right triangle makes it entirely possible to find the length of the hypotenuse. In exactly the same way, before concluding the formula for the square of a side of a triangle, he should realize that knowing the lengths of two sides of a triangle is not enough, that another element is needed.

It goes without saying that the teacher can demand of the pupil a completely conscious mastery of the study material only if the lessons develop, with enough thoroughness and depth, all the factual and conceptual content of this material.

If, however, the teacher knows how to present a mathematical fact, not by itself, but as a part of a chain of organically interrelated facts, he has the right to assume that the pupil too will gradually learn and subsequently will be in a position to approach the study and analysis of mathematical theory.

Soviet teachers have long known that if a topic is to be thoroughly revealed, one must pose a system of questions on the topic to the pupils that elucidate the essence of the concepts and facts contained in the topic. They also know that it is especially important that these questions train the pupils to draw independent conclusions from the theory they have learned and to apply it on their own as they solve simple theoretical and practical questions.

The Systematic Nature of Presenting and Mastering the Material

1. Mathematics as a science is a well-constructed system of scientific facts whose theory may be presented only in a strict logical sequence. Mathematics as a subject of study likewise cannot be presented without retaining the basic requirement—the observance of complete logical sequence. Disturbing this requirement would entail not only digressing from the scientific principle but also such a deeply rooted distortion of the subject that it would simply cease to be a subject of study. The school mathematics course should be a strictly contained system of information from the field of mathematics. It would be unthinkable to study or to teach a school mathematics course in which, for instance, terms would be used before being defined, or in which one had to make references to still unproved propositions to prove a hypothesis, and so forth. A carefully thought-out curriculum and a suitable textbook save the systematic nature of the teacher's presentation of the mathematics course from this type of distortion. The defects that might appear in the curriculum and the textbook would, of course, be noted and eliminated in due course.

2. The teacher who attempts to be systematic in the construction of a method of teaching a section or unit of the course finds himself in a different position. To achieve his aim he has to think out and prepare (a) a system of the distribution of all the material relative to the section or unit; (b) a system of questions that he will have to ask when

presenting the new material and when reviewing and expanding it; (c) a system of exercises for the pupils to do in class and at home. The result of the teacher's work will be systematic if he finds a proper order and logical sequence of ideas such that each new idea is prepared naturally and follows from the preceding one, and if the questions and exercises are arranged in a definite sequence. The teacher, as he elaborates the topic methodically on his own, will have to work hard at first, but later he will have only to make some corrections and improvements on the basis of his experience. If his selection of a system of exercises is helped by good workbooks with problems and examples, he achieves a systematic teaching of theory based primarily on his own knowledge and experience. But the element of creativity in this work will give him great satisfaction and inner joy.

3. A most carefully planned systematic presentation of the material is utterly necessary for the pupils to master new material consciously and thoroughly. Only gradually, in a definite system effecting stratification, by adding new knowledge to previous logically and organically related knowledge, can a sufficient mastery of a subject be ensured. The pupil should use what is acquired, recognized, and known as a basis for understanding new material. In understanding newly mastered material, his thought seeks analogies, examples for generalizations, sources of the appearance of new facts, and methods of substantiating them.

Analysis of the reasons for the pupils' lack of comprehension of some concepts or facts in a theory often leads to the conclusion that in learning these facts, the pupils did not observe that system, that sequence which is the only one permitted and regular in a given case, and whose destruction could not help bringing the pupils to a misunderstanding of theory and to vagueness in their minds.

The systematic structure of the material to be presented should also take into consideration all the psychological and logical elements of the process of the pupils' mastery of new material. Here are some examples.

Example 1. For the unit on "Similar Triangles," the teacher can prepare the following plan.

a. Similarity: Observed in nature (leaves of trees, petals of flowers, the wings of butterflies of one species); observed in architecture (the plan of the facade of a building and the facade itself);

observed in surveying land (the map of a locality and the locality itself); obtained through photography, etc.

b. Establishing the first definition: "Two rectilinear figures are called similar if they have an identical form but different dimensions."

c. Making this definition more precise: For identity of form both figures must have the same number of sides and angles and correspondingly equal angles; the fulfillment of this condition alone is not enough, however--complete identity of form exists only if the sides of one polygon are proportional to the sides of the other.

d. Establishing the method of constructing the conditions to prove that the corresponding equality of the angles of two quadrangles, pentagons, and in general, figures of n sides where $n > 3$, does not entail the proportionality of the sides. It suffices to draw an arbitrary pentagon ABCDE, draw from some point A' in the plane the segment A'B' parallel to side AB and equal to, say, $\frac{1}{2}$ of side AB, then from point B' the segment B'C' parallel to side BC and equal to, say, $\frac{2}{3}$ of side BC, and finally, from point C' the segment C'D' parallel to side CD and equal to, say $\frac{3}{4}$ of side CD. If, we then draw half-lines D'E' and A'E' from points D' and A' so they are parallel to sides DE and AE respectively, until they intersect each other at point E', a pentagon A'B'C'D'E' is obtained whose angles are correspondingly equal to the angles of pentagon ABCDE, but whose sides are not proportional to the sides of this pentagon, since

$$\frac{A'B'}{AB} = \frac{1}{2}\,;\quad \frac{B'C'}{BC} = \frac{2}{3}\,,\quad \frac{C'D'}{CD} = \frac{3}{4}\,.$$

The pupils should notice that whenever the given polygon has five sides, the corresponding lengths of only three sides of polygon A'B'C'D'E' may be made to order, since the lengths of the last two sides will already be determined by their direction.

A similar situation may be observed in constructing a quadrangle A'B'C'D' whose angles are correspondingly equal to the angles of a given quadrangle ABCD, but whose sides are not proportional to the sides of this quadrangle; again it appears that we cannot specify the lengths of the last two sides.

e. A different conclusion must be drawn when using the same method to construct triangle A'B'C' whose angles are correspondingly equal to the angles of a given triangle ABC. In this case, after having specified the length of side A'B', we have to limit ourselves to this, since the lengths of sides B'C' and A'C' are already determined by their direction, and the point C' will appear as the point of intersection of the half-lines B'C' and A'C'.

f. The question arises whether the lengths of the sides of triangles ABC and A'B'C' stand in some mutual relationship. The theorem is given and proved.

> Theorem 1. If the angles of one triangle are correspondingly equal to the angles of another triangle, then the sides of these triangles are proportional.

g. Having written theorem 1 in the form

$$\frac{\angle A' = \angle A,\quad \angle B' = \angle B}{\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC} = \dfrac{A'C'}{AC}\ ?}$$

the teacher directs the pupil's attention to the fact that the condition of this theorem consist of two independent relations and that from this system of relations there follows each of the two independent relations included by the theorem, so that the theorem has essentially two conclusions. Under the teacher's guidance, the pupils should compose a formula and prove the two inverse theorems.

Theorem 2. $$\frac{\angle A' = \angle A;\quad \dfrac{A'B'}{AB} = \dfrac{A'C'}{AC}}{\angle B' = \angle B\ ?}$$

Theorem 3. $$\frac{\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC} = \dfrac{A'C'}{AC}}{\angle A' = \angle A;\quad \angle B' = \angle B?}$$

h. Theorems 1-3 are formulated as three indications of the similarity of triangles.

i. Application of the established theory to the solution of practical problems.

Example 2. For the unit on "Properties of the Exponential Function," the following plan may be composed.

a. Re-establish in the pupils' minds the following fact (which they know from arithmetic) from the unit on "Multiplying Fractions." In multiplying a number by a proper fraction we obtain in the product a number smaller than the multiplicand; in multiplying a number by an improper fraction, not equal to 1, we obtain in the product a number greater than the multiplicand.

b. This proposition underlies two propositions concerning positive integral exponents: When a positive fraction is raised to a positive integral power, its character (i.e., of being a proper or an improper fraction) is not changed; and for a positive base greater than 1, the larger the exponent, the larger the power and for a positive base less than 1 the larger the exponent, the smaller the power.

The two propositions in (b) are first examined using examples, then proved generally, using the arithmetical fact indicated above.

c. The two propositions in (b) are expanded to cover the cases of the positive fraction, and the negative (integral and fractional) exponent, for which examples are first examined, then proved generally, utilizing the properties of numerical inequalities.

d. Construct, by points, graphs of the functions

$$y = 2^x, \quad y = (\tfrac{1}{2})^x, \quad y = 3^x, \quad y = 10^x,$$

and use the established properties of exponential functions to make the course of these graphs more precise.

e. Using the graph of the exponential function as an example, establish the concepts of the monotonic change of a function, the asymptotic approximation of a curve to some axis, and the unlimited growth of a function. Compare the graph of the function 2^x to one of the function x^2.

f. Using the notation of the conditions of the two general theorems established: If $a > 0$ and $a \lessgtr 1$, but $\alpha > 0$, then correspondingly $a^\alpha \lessgtr 1$; if $a > 0$ and $a \lessgtr 1$, but $\alpha < 0$, then correspondingly $a^\alpha \lessgtr 1$; if $a > 0$, $a \lessgtr 1$, $\alpha > \beta$, then correspondingly, $a^\alpha \lessgtr a^\beta$. Ask the pupils to compose formulas and prove (by the indirect method) all possible converse theorems.

g. Ask the pupils the following system of questions:

1. Are the powers

$$\left(\frac{3}{4}\right)^{2/5} \text{ and } \left(\frac{6}{5}\right)^{-7/10}$$

numbers greater or less than 1?

2. What conclusion may be drawn

   a. About the exponent α if

$$\left(\frac{3}{5}\right)^{\alpha} = \frac{9}{4}\,;\ \left(\frac{7}{8}\right)^{\alpha} = \frac{5}{6}\,;$$

   b. About the positive base a if

$$(a)^{3/8} = \frac{5}{9};\quad (a)^{-0.4} = 2.7\ ?$$

3. Which of the two powers is greater:

$$\left(\frac{5}{6}\right)^{3/8} \text{ or } \left(\frac{5}{6}\right)^{7/8}\ ;\ \left(\frac{9}{8}\right)^{4/5} \text{ or } \left(\frac{9}{8}\right)^{6/5}\ ?$$

4. What conclusion may be drawn

   a. About exponents α and β if

$$\left(\frac{2}{3}\right)^{\alpha} < \left(\frac{2}{3}\right)^{\beta}\ ;\ 2.3^{\alpha} < 2.3^{\beta}\ ;$$

   b. About the positive base a if

$$a^{3/8} < a^{9/8};\ a^{-4/5} > a^{6/5}?$$

h. While the pupils are studying the unit on the exponential function, give them exercises related to it, including construction of a graph of an exponential function, and solution of exponential equations analytically and graphically.

Example 3. For the unit on regular polygons, the following plan may be proposed.

a. The concept of the regularity of the polygon. Is the equality of the sides of a polygon the result of the equality of its angles, or, conversely, is the equality of the angles a result of the equality of its sides? An exception: the triangle.

b. The relation: Regularity is identical (equivalently, adequately) to a system of two facts--to the equality of angles and the equality of sides, which, for all polygons except triangles are mutually independent.

c. The construction of regular polygons, using a circle divided into equal parts. Convex and concave (star-like) polygons.

d. The existence of a point simultaneously equidistant from all vertices and all sides of a regular polygon. Using equality of the angles and sides in proof.

e. In the case of the convex quadrangle (any one), the equality of its angles entails the existence of the center of a circumscribed circle, and the equality of its sides entails the existence of the center of an inscribed circle. Where both relations exist, the centers coincide.

f. The similarity of regular polygons of the same type. The coefficient of similarity. Proportionality of like linear elements (radii, apothems, diagonals, perimeters).

g. The division of a circle into equal parts using a compass and ruler. Gauss' theorem (without proof).

h. The calculation of the sides of regular polygons (convex and starlike).

i. The approximate division of a circle into equal parts.

j. Historical information.

k. Questions on the topic:

1. Under what conditions does there exist a point equidistant from n points in the plane; is this a necessary condition?
2. Does an irregular polygon have a center?
3. How many ways are there to find the center of a regular polygon? Can this be done using the construction of diagonals?
4. How many elements determine a regular polygon?
5. How many regular polygons (convex and star-like) may be obtained by dividing the circle into 5, 8, 10, 12, 15 equal parts?
6. How can one construct the side of a dodecagon, an icosagon?
7. Can one construct, using a compass and straightedge, a regular figure with 64 sides; with 9 sides?

The Purpose and Ideological Intent of the Lesson

1. In thinking over a lesson to be presented, the teacher should

imagine precisely what the aim or the designation of the lesson is, what new concepts or ideas it reveals to the pupils, what new knowledge, abilities, and skills they shall obtain. The conceptual content, the interest, of teaching must be juxtaposed against a dry, formal internally unenlightened communication of information.

Underlying the school mathematics course are ideas such as the principles of calculation (one of the greatest triumphs of human culture), algebraic symbolism (at a higher level, a useful apparatus for performing and noting mathematical research, as well as a language understood by any educated man), the method of equations (one of the major sections of the school algebra course, which determines the significance of the course as a subject that develops methods of solving and investigating many problems by utilizing functional relationships among quantities), the concept of a variable (including general properties of real variables observable in phenomena of the material world--the quantitative character of the mutual connections of changes occurring in it), the concept of a limit and the method of limits (information of the essence of mathematical analysis), the concept of size and its measurement (commensurable and incommensurable quantities, relations between rational and irrational numbers), the idea of axiomatic construction and the deductive exposition of science (the system of geometry).

This short list of the fundamental ideas of the school mathematics course illustrates the rich idea content of his course; each teacher should strive to develop this content as fully as possible and to see that the pupils master it thoroughly.

2. The teaching of mathematics in the Soviet school should use any means at its disposal to develop a dialectical materialist worldview in the pupils, who are the future builders of communism. This can be done both in connection with the establishment of individual facts (introduction of the field of numbers reflecting the relationships among objects of the material world; establishment of concepts of geometric forms as a result of abstraction from the representations of physical bodies; familiarization with axioms as statements formulating real properties of objects in the material world), and especially in lessons devoted to certain general conclusions (development of the concept of number; axiomatics of arithmetic and geometry; ideas of non-Euclidean

geometry, which refute the idealistic teaching of the a priori nature of spatial conceptions). But it goes without saying that the teacher's activity in developing the pupils' world-view should not be limited to individual lessons or even to a series of lessons. Illuminating a problem in its historical development, revealing the role of mathematics as a means of investigating the laws of change in the material world, choosing appropriate problems illustrating this role of science in the investigation of nature and the search for means of furthering it--all of these serve the purpose, too.

Observance of the Scientific Principle

Individual branches of the school mathematics course (arithmetic, algebra, geometry, trigonometry) as subjects of study do not, of course, reproduce the corresponding sciences, but the level of teaching each topic should correspond to the state and treatment of it in sciences. Teaching conducted on this level leads the pupil into the field of the corresponding science, into the realm of scientific concepts, ideas, and methods, creating a firm foundation for his knowledge, widening his mental horizon, and substantiating his world-view.

To implement the scientific principle during the teaching process, the teaching must be conducted at a sufficiently high theoretical level. Definitions of concepts and formulations of axioms and theorems should be given with impeccable precision, with a proper revelation of their essence; proofs of statements should be conducted as rigorously as is possible at the given level of school instruction.

The possibility and necessity of observing the scientific principle are based, first, on the fact that scientific proof and faultless logical argument further the initial understanding of the question; however, an unscientific proof, i.e., a proof that is not rigorous, insufficiently substantiated, gives the pupils only surface knowledge--to relate uncritically to argumentation, to accept as proved what was not substantiated, to agree easily, to master the material formally, externally. Facts that cannot yet be proved at a given stage should be accepted without proof.

In particular, among mathematics teachers there is a widespread mistaken notion that arithmetical facts can be rigorously proved only with the use of letter notation; in fact, any argument using numerical examples, as long as there is no reliance on the individual properties of the numbers used, has the full force of a proof. Thus, in fifth grade, it is no trouble to prove the two statements that are the basis for the conclusion of the criteria of divisibility, and even a further series of statements from later sections of arithmetic as presented in the fifth and sixth grades.

Here are some examples of the violation of the scientific principle.

a. In the definition of the concept of the prime number, in the phrase "is divisible only by 1 and by itself" the most important word--"only"--is omitted.

b. In the definition of the concept of proportional quantities, in the phrase "is increased or decreased so many times" the most essential words--"so many times"--are omitted.

c. In checking a proportion by establishing equality between the product of the means and the product of the extremes, reference is made to the direct theorem: "If a proportion is correct, the product of its extremes equals the product of its means," but one must refer to the converse theorem, "If four numbers are such that the product of two of them equals the product of the other two, a proportion can be made of these numbers, taking the first two numbers as the extremes or the means of the proportion and the other two numbers as its means or extremes."

d. In composing a quadratic equation, given its roots, calculation is based on the property of the roots of a quadratic equation, this property being expressed by the theorem: "If α and β are the roots of a quadratic equation $x^2 + px + q = 0$, then $\alpha + \beta = -p$, and $\alpha\beta = q$," but one must use the theorem: "If the numbers α and β satisfy the equations $\alpha + \beta = -p$ and $\alpha\beta = q$, then these numbers are the roots of the quadratic equation."

e. The definition, "An equation is an equality that is true for certain values of the unknown," is inadmissible since it does not include equations having no roots. This remark also refers to the definition of the concept of a system of equations.

f. If in the proof of the identity of an inequality one starts from the proposition that the inequality being considered is true for all allowable values of the letters it contains, then, approaching an obvious inequality, one must prove by the converse method that the original inequality is the result of this obvious (deliberately true) inequality; this conclusion will be valid whenever all operations performed in the direct transition are reversible.

g. The definition of an irrational number as a root of the nth degree from an incomplete nth degree of some number does not correspond to the scientific interpretation of an irrational number.

h. A shining example of a quite unsubstantiated argument is the explanation, given in several texts (Kiselev's, among others), of the solution of a system of linear equations; this explanation comes down to an indication of rules by which the solution of a system can be found and contains not even the most elementary theory of the problem.

i. In defining the concept of the locus of points, there is no indication that the locus of points that has property α should contain all points having this property and only these points, or the words "and only these points" are omitted.

j. In formulating the axiom of parallel lines, it is not stressed that it is not establishing the existence of a line b passing through a given point A and parallel to a given line a, but the uniqueness of this line b.

In presenting mathematics scientifically, logical elements of arguments, which are of great educational value, acquire special significance. These elements are: (a) the ability to establish and formulate accurately all conditions and the conclusion of the statement being proved, and then to form a course of proof in the form of a short but expressive and visual notation which, beginning in the upper grades, becomes increasingly symbolic; (b) the ability to compose the inverse or contrary theorem for a given theorem; (c) the ability to make a correct logical division of a class into subclasses, to pick out types from a class and distinguish type features from class features, to distinguish a feature from a definition, to establish whether a condition is necessary or sufficient.

Training in logic, although given in all school disciplines, may be done with the most success in mathematics lessons due to the specific character of mathematics. For in studying mathematics, the logical structure of an argument can be revealed most vividly. The mathematics teacher should carefully think through each proof presented to the pupils and should present it irreproachably. In the proof, there should be no omissions that would destroy its rigor, even though the pupils might not notice the omissions. The pupils' reasoning might (and should) involve mutual criticism of the chosen paths and methods of proof, as well as of logical gaps and flaws that occur.

These requirements should be applied not only to new proofs but also to the pupils' oral and written answers, which should be evaluated in relation to both their content and especially their fomulation, which very often allows one to judge the pupils' flaws in thinking.

Training in logical thinking, however, should be done gradually, becoming more intensive as the upper grades are approached, in which the level of development, the interests, and the need for proper arguments are conducive to it.

Teachers fall short of fulfilling these requirements (requirements which must be satisfied in the methodology of mathematics teaching for this methodology to promote logical thinking) in the following ways:

1. By not considering it their duty to isolate all points in the statement and conclusions of theorems precisely and to make a notation of the course of the argument, as well as to demand it of the pupils in their classwork or homework.

2. By permitting a number of logical defects in their explanations (insufficient rigor of arguments, incompleteness or insufficiency of argumentation, lack of substantiation for generalizations and analogies, incorrectness of classification, etc.) and, more often, by not explaining or eliminating these defects in the pupils' written and oral answers.

3. By reducing their demands on the pupils in solving geometry problems and the construction of geometric figures when substantiation is required.

4. By reducing especially their standards for substantiation of algebraic and trigonometric theory, and by making almost no demands for justifying the rules of arithmetic.

5. By permitting errors in the notation of the pupils' work, both in class and at home because of a lack of systematic checking; disorderly (and consequently illogical) notation is observed in many notebooks; strict, sequential (both written and oral) expositions of arguments have

not been required of pupils; they are not dissatisfied with their disordered or insufficiently ordered thoughts.

In view of the cardinal importance of noting the conditions of theorems and the course of their proof, we shall present some examples of this notation as used in the sixth, seventh, and tenth grades.

Example 1. Theorem: "The external angle of a triangle is greater than either nonadjacent internal angle."

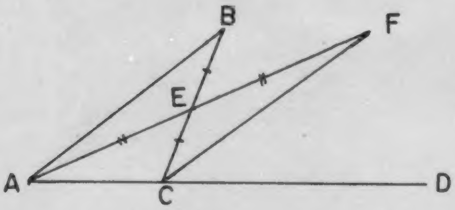

Figure 1

$\angle BCD$ is the external angle of $\triangle ABC$

$\angle BCD > \angle B$?

Complete notation of the proof

Construction: $BE = EC$; $EF = AE$; we connect points C and F

1. $\triangle CEF \cong \triangle AEB$
   - $\angle AEB = \angle CEF$ (vertical angles)
   - $AE = EF$ (by construction)
   - $BE = EC$ (by construction)

2. Therefore, $\angle BCF = \angle B$.

3. $$\left.\begin{array}{l}\angle BCD > \angle BCF\\ \angle BCF = \angle B\end{array}\right\} \angle BCD > \angle B$$

Short notation of the proof

1. $\triangle CEF \cong \triangle AEB$; $\therefore$ $\angle BCF = \angle B$.

2. $$\left.\begin{array}{l}\angle BCD > \angle BCF\\ \angle BCF = \angle B\end{array}\right\} \angle BCD > \angle B$$

As can be clearly seen from these notations, the short notation differs from the complete one in that it omits justifications that are included in the complete form. In the sixth grade (even in beginning the study of geometry) the short form is more convenient if the justifications are given orally by the pupils; moreover, the justifications should have content, i.e., they should be verbal references to the points in the conditions that are utilized, as well as the points of the proof and the statements used.

Example 2. Theorem: "The largest angle in any triangle is the one opposite the largest side."

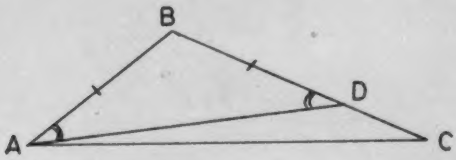

Figure 2

In $\triangle ABC$

$\underline{BC > AB}$

$\angle A > \angle C$?

Short notation of proof

Construction: $BD = AB$; join points A and D. $\angle A > \angle BAD$; $\angle BAD = \angle BDA$; $\angle BDA > \angle C$. $\therefore \angle A > \angle C$.

Example 3. Theorem: "In any parallelogram the diagonals bisect each other at their point of intersection."

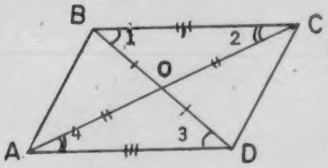

Figure 3

1. ABCD is a parallelogram;
2. AC and BD are its diagonals;
3. AC intersects BD at point O.

$AO = OC$, $BO = OD$?

Complete notation of the proof

1. $\triangle AOD \cong \triangle BOC \begin{cases} AD = BC; \\ \angle 1 = \angle 3,\ \angle 2 = \angle 4. \end{cases}$
2. $\therefore\ AO = OC;\ BO = OD.$

Example 4. The theorem on the criteria for parallelism of a line and a plane.

1. $a \not\subset \alpha$
2. $b \subset \alpha$
3. $a \parallel b$

$a \parallel \alpha$?

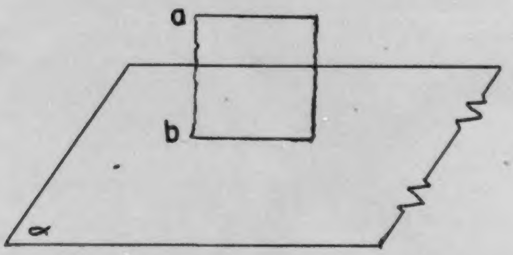

Figure 4

Complete notation of the proof

1. Let a be not $\parallel$ to α.
2. Then a x α at some point K (condition 1) (Point K is not shown).
3. Plane (a,b) x α along line b (conditions 3, 2, 1).
4. $K \subset a$, $a \subset$ plane (a,b); $\therefore K \subset$ plane (a,b); $K \subset \alpha$, $K \subset$ plane (a,b), $\therefore K \subset b$, which contradicts condition 3. Thus $a \parallel \alpha$.

In this notation (suggested for ninth grade), mathematical symbolism is already being used ($\subset$ is the symbol of inclusion in a class or the symbol of belonging, x is the symbol for intersection, $\parallel$ is the symbol for parallelism). References to points in the conditions are in parentheses at the end of steps 2 and 3 of the proof, which indicates how proved statements are substantiated. These points should, of course, be stated in detail verbally by the pupils.

From the above notation one can imagine the degree of rigor required in mathematical argumentation in the upper (ninth and tenth) grades. By doing proofs with such rigor, the pupils can be brought to a full understanding of the substantiated deductive argument. They also will acquire skill in constructing such arguments and in giving the required justification for each step without skipping from one statement to the next or drawing unwarranted conclusions. At this stage the teacher should obtain from the pupil an adequate explanation of each assertion, accompanied by a precise and verbally formulated reference to a point in the conditions or the proof of a statement proved earlier. It was impossible, for example, in the proof given above, to assert that planes (a,b) and α intersected along line b only on the basis that b is their common straight line; it was necessary to prove that these planes do not coincide, due to condition 1.

It is especially important that the pupils notice how clearly all three points of the conditions used were indicated in the last notation (in the proof of the theorem of the parallelism of a line and a plane), and how no condition was included that did not belong in the data. Failure to use any one of the given conditions would mean that it was superfluous, i.e., that it was mistakenly included among the conditions of the theorem, whose inference (conclusion) would thus not depend on this condition. Drawing on a new condition, however, would show that the inference (conclusion) of the theorem did not proceed from the given conditions, was not a result of them, and therefore the theorem in its present formulation would be incorrect.

Accessibility of the Material

The methodology of teaching mathematics, as it exists in theory and is effected in practice, is constructed basically on the principle of accessibility. This is true because all devices of teaching individual topics recommended in this methodology presuppose a definite store and level of knowledge, and take account of the development appropriate to the pupil's age and of his ability at this age to perceive abstract knowledge. However, the problem of creating a methodology of mathematics teaching that would bring the material in this discipline fully within the pupil's grasp has not yet been resolved.

This situation is significantly aggravated because the study material in the present curriculum has still not been arranged so as to correspond in every possible way to the material's comprehensibility to the pupils. Thus many years of experience have brought teachers to the unanimous conclusion that the arithmetic course (especially in parts containing instruction in fractions) is not completely accessible to fifth-grade pupils, and the geometry course, as it has been taught up to now in the sixth grade, is inaccessible to these students. The geometry course is abstract, divorced from reality, and corresponds neither to the pupils' level of development nor to their interests and requirements. These materials deprive the teacher of the possibility of drawing on concrete facts that are familiar to the pupils and on the natural curiosity and inquisitiveness characteristic of children at this age.

Other topics can be cited which entail significant difficulties in teaching, due to their inaccessibility to the pupils. The theory of equivalence of equations in the seventh grade, the theory of measurement of quantities (using the Euclidean algorithm) and the chapter on irrational numbers in the eighth grade, the study of limits and its application in the ninth and tenth grades, and the chapter on complex numbers (in its present exposition) in the tenth grade are a few examples.

But what can be recommended to the teacher who is struggling to make his material clear to the pupils? Above all, he should bring into operation general didactic hints relating to this topic and usually expressed in formulas: "From the near to the far," "From the known to the unknown," and "From easier to harder" -- the best choice and combination of which will be suggested to him by his experience. For mathematics teaching, these general hints may be supplemented by the following ones:

1. "From the particular to the general." The teacher might preface a theory with a group of well chosen examples and problems. Such a method of helping the pupils to understand theory is practiced at all levels of instruction and is highly effective. At times the teacher finds it possible and even expedient to limit himself to the "establishment" of a fact, using examples (criteria for divisibility in the fifth grade, division of a polynomial by a polynomial in the sixth grade,

properties of equations in the seventh grade, finding square roots in seventh grade, etc.), so as to lay the groundwork for some of these topics in the upper grades. We repeat that strict reasoning done with an example, but not using individual properties of this example, has the full strength of proof.

For example, in examining three cases which may appear in solving systems of linear equations, one might begin with solving the systems of linear equations using numerical coefficients and establish by examples that these systems can be defined, undefined, and inconsistent, and then find criteria for a system's inclusion in each of the possible forms.

The conclusion of the formula for solving a quadratic equation by isolating a perfect square on the left-hand side of the equation is first made using a series of examples of equations with numerical coefficients; these equations gradually increase in complexity. After this the conclusion of a general formula will no longer present any difficulty to the pupils.

2. "The more accessible the study material becomes to the pupils , the more they participate in class." It is hard to overestimate the role of making pupils participate in order for them to attain complete understanding of the material being communicated. It is well known to the experienced teachers who become accustomed to using activization of the pupils as a powerful lever in conquering all the "difficult" places in the course. A planned system of questions, directed at revealing the essence of a topic, its nucleus, is an effective way of leading the pupils to a thorough understanding of the material.

For clarifying the possible results of decimal measurement of segments, the following system is suggested:

What number expresses the result of decimal measurement of segment A by segment E if $A = 2.7E$; $A = 3\frac{1}{8}E$; $A = 5\frac{1}{7}E$; segment A is incommensurable with segment E?

3. "The material becomes accessible to the pupils, if the teaching is accompanied by the use of visual devices."

When the teacher foresees or observes in practice that the material as presented is not simple enough, he tries to find methods that might help him to present it visually.

Visual Aids in Mathematics Instruction

Instruction is called visual, if it is based on the pupils' direct perception of objects and phenomena by their sense organs. This study of concrete objects and phenomena is necessary to cross "from lively contemplation to abstract thought." Only by having entrenched in his consciousness a sufficient number of vivid impressions of objects appearing as concrete (accessible to direct perception) images of a concept that is studied will the pupil be in a position to construct the concept, with the help of the act of abstracting, in his mind, and make it something he has mastered, an element of the knowledge gradually accruing inside his head. As an example, the child finally grasps the concept of the number "five" only after repeated observation of groups of homogeneous objects appearing as concrete representatives of this number "five," images accessible to direct perception.

The teacher develops the concept "three-eights" in the pupil's mind, marking off before his eyes three one-eighth portions of a circle, each portion being one of eight equal sectors of the circle. And the teacher explains that $\frac{1}{3}$ of $\frac{1}{4}$ of a circle is $\frac{1}{12}$ of the circle by dividing the circle into four equal sectors and then subdividing each of these sectors into three new, equal sectors, of which there are 3 x 4, i.e., 12, in the whole circle. The same operation, repeated in application to a segment, makes concrete the proposition that $\frac{1}{3}$ of $\frac{1}{4}$ is $\frac{1}{12}$. Frequent repetition of the same device for various pairs of fractions leads by generalization to the conclusion that $\frac{1}{n}$ of $\frac{1}{m}$ of any quantity is equal to $\frac{1}{mn}$ of this quantity; here one abstracts oneself from the kind of quantity of which the operation of division into equal parts is performed; it is necessary only that this quantity permit such division (that it be additive).

All elementary geometric concepts (solid, surface, line, point) are established by examining concrete (physical) solids, surfaces, lines, and points and finding the essential properties that determine them. Although abstract concepts are thus created in each pupil's mind, when the pupil, in all his subsequent reasoning, speaks of the straight line, for example, he will visualize one of those concrete images of the straight line, whose abstraction from individual properties has led him to the concept of the straight line. Where possible, our thinking replaces the abstracted

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concepts with their concrete representations; this fact is very important for substantiating visual devices that teach one to see, in every concrete image of a concept, its essential properties, i.e., these devices replace abstract thinking with concrete thinking.

The examples we have given of the application of visual instruction that aims to create an abstract image do not exhaust the cases in which visual aid acquires a special significance. A very essential role is played by visual aids in the examination of facts and phenomena which permit so-called geometric interpretation, a representation of numbers and relations between numbers, using geometric images--points, segments, lines. It would be harder to become aware of the world of numbers and relations among them if numbers could not be replaced by their geometric representations--segments and points on the axis and in the plane, and relationships between numbers as given in assignments--graphs of these equations. The teacher uses this idea widely in solving arithmetic problems, in composing equations from problem conditions, for geometric interpretation of the solutions of equations and systems of equations, in studying the properties of functions, for illustrating sequences, for representing real and complex numbers, in the study of trigonometry, and in many other cases.

In studying solid geometry, models are inspected, leading to concepts of points, lines, and planes in space, of polygons and round bodies. Here the same process of abstraction occurs as in establishing concepts of points, lines, and surfaces in the beginning of the study of geometry.

But moreover--in studying the mutual positions that lines and planes in space may occupy relative to each other, as well as various properties of these objects formulated in axioms and theorems--visual aids can be sticks, knitting needles, records, i.e., concrete representatives of previously formed concepts. Again, abstract thinking is helped by conceptions arising as a result of observing real objects, having--at the observer's command--a certain relative position in space. In crossing from these tangible visual aids to drawings, the pupils bring their imagination into play and gradually acquire the ability to understand the drawing, to create a true mental picture corresponding precisely to the drawing, and then to reproduce a conception of the required image, without using a drawing. Man needs this capacity for spatial imagination, for mental reproduction of spatial conceptions. It should be inculcated in every way possible as one of the most important aims that every pupil must achieve during his general education.

These, in general, are the fields in which visual instruction, in the broad sense of the word, are applied. The teacher, however, would be making a mistake if he limited himself to these fields. The visual aids should not be drawn upon sporadically, from case to case, from topic to topic, but systematically, and should be made an organic part of the teaching of a subject, underlying a concretely individual method of instruction, beautiful models of which were given by K. F. Lebedintsev, a famous Russian methodologist, in his texts and problem-books. The concrete-inductive method supposes a broader interpretation of visual aids than we have disclosed above. In this broad interpretation, visual aids are generalized to concreteness, which could consist not only in the possibility of directly observing an object or phenomenon, but in construction of general conclusions only after (and on the basis of) examining a series of selected objects the abstraction of whose peculiarities should lead to necessary general conclusions. Thus here, too, visual aids for teaching consist in the accumulation (for the instruction process), of abstract concepts in the imagination, which are derived from the study of concrete material.

The Connection Between Theory and Practice: Exercises in Mathematics

1. The question of the connection between theory and practice, the consideration of which constitutes one of the most essential requirements of pedagogy, has become at present--in the transition to universal polytechnic education--a problem of first importance, which is being studied and developed by the science of education.

Previously the link between theory and practice was understood to mean that the study of theory should be accompanied by the execution of a system of exercises that would promote conscious and thorough mastery of the theory and the acquisition of the necessary abilities and skills. Now a requirement has been added to these basic concepts, which are still valid--to direct mathematics instruction and its applications so that each middle-school graduate would be able to manage his acquired knowledge any time he needs to in practice, in solving a problem, to present the problem in a mathematical form, do the required operations (calculation, construction, investigation) skillfully and rationally, and find a meaningful answer. Such an approach to mathematics instruction, constantly directed toward exposing all means that mathematics may provide for solving practical problems, should enable the pupils to master the methods gradually, more fully and diversely, and to become convinced practically of their wide applicability and force, until finally the pupils will be using them as a matter of course in solving and investigating quantitative questions. Among such mathematical means are devices for direct calculations (precise and approximate) and calculations using tables; methods for doing rational identity transformations, methods for composing and solving equations and systems of equations; methods for investigating solutions of equations and problems with numerical and parametric data; devices for investigating the change of functions and construction of their graphs; methods for solving construction problems; geometric and analytic methods for solving triangles; means of representing spatial figures on a plane; means of graphic solution of equations and systems of equations; devices for computations using the slide rule.

The pupils' utilization of all of these means should be organically connected with the study of theory; it should be not only an illustration of the application of theory to practice, or a proof of the effectiveness of this theory; it should have independent value as a method for the pupils to acquire a system of very important abilities and skills, wholly necessary for their future activity.

The teacher should never relax his attention to this aspect of mathematics teaching, but should strive constantly to fill it with valuable content by reinforcing the connection between theory and practice. Selection of exercises and problems in the text should be a special concern of the teacher. He should always be on the watch for questions for whose solutions he should skillfully—after several preliminary quests, arguments, and considerations involving a certain amount of mental effort--apply certain methods offered by mathematics.

2. Exercises in mathematics should be combined with theory into a single organic whole. They will not only illustrate theory but will

almost always delve deeper into it and supplement it. The teacher should bear this in mind when selecting exercises. At the first stages his aim is to inculcate firm abilities and skills in the content of the chapter being studied. But at later stages, while still attending to this main purpose, he constantly and to a greater measure should strive to impart to the pupils the ability to do exercises unassisted. In this connection, the following requirements should be applied to the methodology of conducting exercises:

a. Practical tasks should be done by the whole class with maximum participation. In other words, the pupils should be active not only in studying theory, but in doing exercises as well. It would seem that teachers are not aware enough of this obvious requirement or do not put it into practice. Often only a few pupils will do the homework assignments or examples and problems the teacher proposes in class, while the rest sit in silent agreement. There is no lively discussion, which would not only uncover any mistakes but would also criticize the methods of solution, show the most logical ones, and establish--with the teacher's aid--useful general conclusions that should lay the foundation for further practice.

In addition, the teacher, drawing on the pupils' activity, should re-examine the various aspects of the theory being studied by using examples and, where possible, should deepen and even expand the examples. For instance, in the unit on algebraic fractions as the teacher establishes the most rational means of making transformations and performing operations on fractions, he also should explain the set of allowable letter-values and the identical character of these transformations and operations.

In solving systems of numerical linear equations, the existence of three groups of these equations--defined, undefined, and inconsistent--has been established. In solving problems using what has been established and seeking new loci of points, the teacher may go beyond the framework of the topic. He may be impelled to do so by the pupils' natural enthusiasm for interesting problems and their desire to find and apply new examples of loci; the pupils' activity in this case should be encouraged.

In solving second-degree equations by using examples, one may establish all properties of the integral rational second-degree function with real coefficients: the criteria for its introduction into the field of real numbers; the concept of its roots; the relationship between its roots and coefficients and the check of its roots based on this; its increase and decrease; the attainment of its largest and smallest values.

We gave these examples only to explain our ideas. However, almost any topic can be made more profound and concrete through the execution of exercises relating to it.

b. Written exercises should definitely be alternated or combined with oral and semiwritten exercises.

As is well known, the principle of applying oral counting in calculations is highly regarded in the teaching of arithmetic. Unfortunately, this very valuable principle has found too small a place in algebra and especially geometry. Nevertheless some of the calculations done in solving algebraic examples can and should be given orally, at least in part (semiwritten exercises). A teacher's requirement that all stages of calculation be written out in detail should be recognized as ill-advised and even harmful. Oral calculations save time and enable the pupils to find the shortest and most correct ways to solve examples, as well as to check their work.

Oral calculation can be especially widely used in working on identity transformations of rational and irrational expressions. But it also is useful in several other sections.

The product $(a^3 - 2a^2 + 5a + 1) \cdot (4a^2 - 6a + 3)$ of two polynomials set up in decreasing powers of the same letter can be found by the method of "grouping the elements containing the same power of the principal letter": $4a^5 + (-6 - 8)a^4 + (3 + 12 + 20)a^3 + (-6 - 30 + 4)a^2 + (15 - 6)a + 3 = 4a^5 - 14a^4 + 35a^3 - 32a^2 + 9a + 3$, where the coefficients are found and calculated orally so that the result is written immediately.

For proving the identity $a(b + c)^2 + b(c + a)^2 + c(a + b)^2 - 4\,abc = (b + c)\,(c + a)\,(a + b)$, the pupil must orally establish the coincidence of the coefficients of identical powers of some letter taken as the main one. For example, the coefficient of a^2 on both the left and right sides will be the expression $b + c$; of a, the left expression will be $(b + c)^2 + 2bc + 2bc - 4bc$, and the right expression will be $(b + c)(b + c)$, identical with that on the left; the free element on the left will be the expression $bc^2 + b^2c$, and on the right, $(b + c)bc$, which is identical to it.

For a great many examples in algebra and a number of problems in geometry, one may establish orally only the plan of solution, the most expedient course, which is a good basis for sounding out the pupils' knowledge and their quick-wittedness and for instilling this ability in them. This type of oral exercise, indicating only the path of solution, without fulfilling it completely or bringing it to an end, permits the pupils to survey, in the shortest length of time, all those examples and problems that are either analogous to models already examined or are insignificant variants of them. In other cases the general course of solving an example or problem might be examined orally before it is assigned for homework, with a view to facilitating and accelerating completion of the assignment. Here the teacher should make every effort to see that this preparation for solving an example or problem be done with the efforts of the whole class, not individual star pupils. Star pupils should supply aid only when the plan of solution does not occur to the average pupil or when this plan can be perfected.

c. When the pupils solve examples and problems, one must be very attentive to any expression of their independence, initiative, and elements of creativity and must awaken in them, as far as possible, the urge to find the most rational ways of solving a problem.

Inculcating these qualities in the Soviet pupil should be the object of the teacher's constant care and should constitute a definite part of his work in preparing for a lesson. The teacher may select appropriate material which opens the field for the pupil's independent thought, for his activity in searching for the best, most expedient ways to solve a problem. This material can include not only practical problems but also--on the order of exercises--theoretical ones. This type of work by the teacher enables the pupils to solve theoretical and practical problems by themselves and is the best way of attaining the fundamental aims of mathematics teaching.

Let us now cite several examples for whose solution devices more rational than those usually used by pupils may be applicable.

To simplify the fraction

$$\frac{\frac{5}{4} - 3 - \frac{4}{3}}{-4 + \frac{2}{5} - \frac{3}{2}},$$

its numerator and denominator are multiplied by the common denominator, 60, of all fractions present in the numerator and the denominator of the given complex fraction; it thereby becomes $\frac{75 - 180 - 80}{-240 + 24 - 90}$.

To factor the expression $a(b + c)^2 + b(c + a)^2 + c(a + b)^2 - 4abc$, it suffices to remove the parentheses and arrange the terms of the polynominal obtained by decreasing powers of some letter, say a: $(b + c)a^2 + (b + c)^2 a + bc^2 + b^2c$ and then group these terms in the order in which they are written: $(b + c)(a^2 + ab + ac) + bc(b + c) = (b + c)(c + a)(a + b)$.

To simplify the fraction

$$\frac{x^3 - x^2 - x + 1}{x^4 - x^3 - 3x^2 + 5x - 2},$$

the pupil rewrites the numerator as $(x - 1)^2(x + 1)$ and must find the common factor of the numerator and denominator by dividing the denominator by one of the factors of the numerator $(x - 1$ or $x + 1)$.

To calculate the sum

$$\frac{a^3}{a - 1} - \frac{a^2}{a + 1} - \frac{1}{a - 1} + \frac{1}{a + 1},$$

it is expedient first to combine the fractional terms having the same denominator and to perform operations on the terms of each group separately:

$$\frac{a^3 - 1}{a - 1} - \frac{a^2 - 1}{a + 1};$$

then the fractions obtained may be reduced and calculation may be done:

$$(a^2 + a + 1) - (a - 1) = a^2 + 2.$$

In precisely the same way, to solve the equation

$$\frac{1}{x + 1} + \frac{1}{x - 1} + \frac{1}{x + 2} + \frac{1}{x - 2} = 0,$$

it is advisable first to replace the first two terms in the left part by their sum, then replace the second two terms by their sum, bringing the equation to the form

$$\frac{2x}{x^2 - 1} + \frac{2x}{x^2 - 4} = 0;$$

then, factoring the left part of the equation,

$$2x\left(\frac{1}{x^2 - 1} + \frac{1}{x^2 - 4}\right) = 0,$$

find the roots: $x_1 = 0$, $x_{2,3} = \pm \frac{\sqrt{5}}{2}$.

To solve the equation

$$\frac{x - 1}{x - 2} + \frac{x - 6}{x - 7} = \frac{x - 5}{x - 6} + \frac{x - 2}{x - 3},$$

it is advisable first to exclude from each fractional element its whole part:

$$1 + \frac{1}{x - 2} + 1 + \frac{1}{x - 7} = 1 + \frac{1}{x - 6} + 1 + \frac{1}{x - 3},$$

writing the equation in the form

$$\frac{1}{x - 2} + \frac{1}{x - 7} = \frac{1}{x - 6} + \frac{1}{x - 3},$$

then, guided by the denominators, to group the terms thus:

$$\frac{1}{x - 7} - \frac{1}{x - 6} = \frac{1}{x - 3} - \frac{1}{x - 2},$$

after which the given equation comes down to the equation

$$(x - 3)(x - 2) = (x - 7)(x - 6),$$

having the root $\frac{9}{2}$.

> The sum of the first three terms of an arithmetic progression is equal to 60. If we add 2.5 4, and 7 to these numbers, respectively, three sequential elements of a geometric progression are obtained.

To solve this problem it is best to designate the three unknown terms of the arithmetic progression as $x - d$, x, $x + d$.

First we find that $3x = 60$, $x = 20$, and then compose the equation $(20 - d + 2.2)(20 + d + 7) = (20 + 4)^2$, that is, the equation $(22.2 - d)(27 + d) = 24^2$. Its roots are $d_1 = -7\frac{4}{5}$ and $d_2 = 3$. This method of designating the terms of an arithmetic progression simplifies calculation appreciably.

d. In homework assignments one should include, as far as possible, a part demanding independent work of the pupils. To complete this part, he cannot limit himself to using knowledge that has been communicated to him.

On the contrary, the assignment will evoke the pupil's interest and he will strive to do it without fail and in the best manner if it contains questions whose answers are not implied or self-suggesting, but which he must think about if only a little, and consider.

e. The content of problems given for practical activities in mathematics should illustrate the value of mathematics as a discipline that gives man the abilities to solve many problems he faces in his practical activity. In this area mathematics should come into contact with adjacent disciplines--mechanics, physics, chemistry, astronomy, geodesy, technology.

Instruction as a Means of Character Development

Most important of the principles which underlie Soviet didactics and methods of teaching in all disciplines is the principle of developmental instruction, the essence of which is developed in the Soviet theory of education. In its application to the teaching of mathematics, this principle is realized by fulfilling the following requirements:

1. Like other school disciplines, mathematics instruction should have as one of its basic aims the development in pupils of a dialectical-materialistic world-view, assuring only a correct, scientific understanding of all surrounding reality, all phenomena occurring in nature and in society. This world-view should be formed in the pupils' consciousness as a result of a gradual, systematic illumination of the laws of dialectics in mathematical examples, organically combined with instruction and naturally explained by these laws (part 4 of this section).

2. Mathematics instruction should be conducted on a level and by methods so that the pupils can develop mentally, can acquire an interest in and a love for the work of the mind, can understand the significance of science and value it as a triumph of the human mind--as the result of the combined forces of people who have freed mankind from delusions and falsehoods and who have directed it to a better future. Such instruction slowly but surely transforms the schoolchildren, who are obliged only to the teacher's explanations and to do assignments, into effective pupils, who are always in an active, creative frame of mind, interested in all or most of the disciplines they study, mastering material, often going far beyond the curriculum's framework, taking a most active part in clubwork--in general, it transforms them into Soviet pupils who bear this

famed name with honor and dignity.

It is in this field of the teacher's activity that the instruction he gives is inseparably linked with character-building, so that one is organically connected with the other, one is imperceptibly transformed into the other. This type of developmental instruction provides the teacher with a vast range for his own creativity and improvement. His personal example, his dedication to science, the inspiration he shows during lessons and in his activity generally—all of these influence the pupil inexpressibly and are an effective force in training the pupil in the way noted above.

3. Mathematics instruction should help the pupils to form good habits of logical thinking and to master structured speech properly—logically and grammatically.

4. Geometry instruction has its own developmental task, especially related to the content of this subject: to develop in the pupils correct spatial representations and a lively spatial imagination, which are necessary for both the study of geometry and related subjects and for their further practical activity. The teacher should work especially carefully to develop this aptitude in the pupils--highly valuable for every educated person-- constantly perfecting it and diversifying its devices and applications. The pupil who does not develop the proper aptitude for spatial imagination cannot be regarded as making normal progress in mathematics.

5. A very important means of character-building in mathematics instruction is in the teacher's hands--the proper selection of problems that will be instructive in establishing the connection between mathematics and related disciplines.

6. The study of mathematics in the Soviet school, i.e., under the Soviet methodology of teaching and the demands of the pupils involved in this methodology, is beneficial in developing the most valuable qualities of the Soviet man. Their will-power is trained, along with perseverance in attaining a goal fixed with determination, assiduity and persistence in overcoming obstacles; constantly—over the whole period of mathematics study--the pupils are acquiring elementary creative habits, the ability to ask, investigate, and resolve a question on their own.

Forms and Methods of Mathematics Instruction

Forms of Mathematics Instruction

In every school discipline, instruction takes place both inside and outside the classroom. We shall examine the forms taken by mathematics instruction in the classroom situation. There are three fundamental ones: the heuristic form, the laboratory form, and the lecture form. We shall begin by developing the essence and content of each of these forms separately. One must not forget, however, that in his teaching the instructor may alternate among these forms, combining them to conform with the general tasks or individual goals he has set for himself.

The Heuristic Form of Instruction. To make the pupils participate actively, for teaching them how to trace a means to solve a given problem altogether consciously, to use this means confidently until the goal has been reached, the heuristic form of instruction is the most expedient. The heuristic method consists of a sequential system of expediently composed and distributed questions to which pupils give answers within their capacity, gradually revealing the essence of the concepts introduced and the facts thus communicated. It goes without saying that the efficacy of this method depends primarily on how logically and methodically the system of questions is composed and how adeptly the teacher guides the entire course of the heuristically conducted lesson.

In the section on Principles of Mathematics Instruction we indicated that to create the necessary psychological situation in the classroom, i.e., a mental tension whose discharge would result in the pupils' active participation, one must begin the communication of new material with a statement of the question. This anticipatory element of a lesson devoted to describing new material is an organic part of it and is inseparable from it. The system of questions on the topic should include this part.

Example 1. Topic: "The theorem of the external angle of a triangle."

The teacher familiarizes the pupils with the concept of the external angle of a triangle and then, directing their attention to Figures 5 and 6, conducts the lesson with the following system of questions.

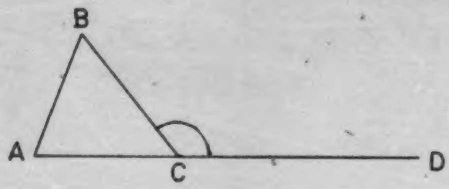

Figure 5

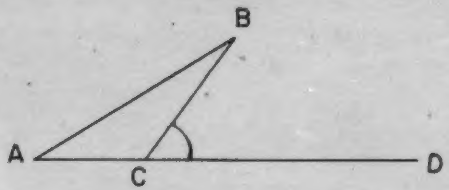

Figure 6

1. Compare the size of the external angle BCD in Figure 5 with the internal angles B and A, which are not adjacent to it.

Angle BCD as an obtuse angle is greater than either of the acute angles B and A.

2. In Figure 6? It seems that here too external angle BCD is greater than either of the angles B and A.

3. Is it possible to have a case in which both the external angle and one of the internal angles are obtuse?

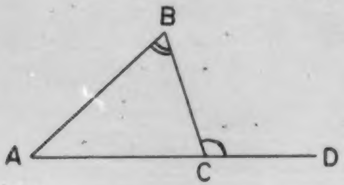

Figure 7

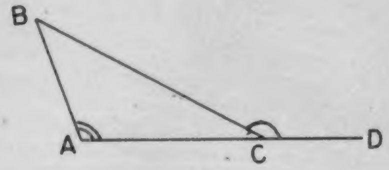

Figure 8

4. What conclusion can be drawn about the sizes of the external angles in Figures 7 and 8?

It seems that here too the external angle is greater than either of the internal angles.[1]

5. Is it possible to have a case in which the external angle is acute and the internal angles obtuse?

No. Lines AE and CB and lines BF and AC do not intersect (Figure 9).

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[1] Measuring with a protractor is permitted.

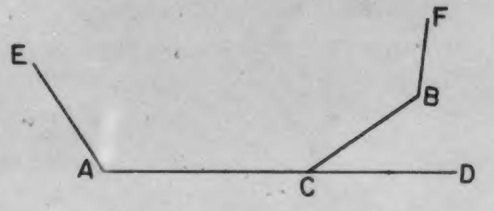

Figure 9

6. What conclusion may we draw from observation?

In a triangle the external angle is greater than either of the nonadjacent internal angles.

7. But is this really so? Can we be sure that each external angle of any triangle has this property? How can we verify this?

This must be established by argument, that is, proved.

8. How will we prove this property we have discovered about the external angle of a triangle? Evidently we shall have to compare it somehow with each of the internal angles. Let us begin, for example, with angle B. Thus, we want to prove that ∠BCD > ∠B. But what does it mean when we say that one angle is "greater" than another?

That means that angle BCD can be divided into two angles, one of which is equal to angle B, so that angle BCD will be the sum of two angles--an angle equal to angle B and some other angle which is the difference of the angles BCD and B.

9. True, but how can we do this division (partition) of an angle into the two angles we need? The Greek geometer, Euclid, whom we have already mentioned, has answered this question. In his Elements he drew segment AM through apex A of the triangle and through the mid-point M of the opposite side BC, and on its extension through the side of point M he marked off segment ME equal to segment AM. He then joined points E and C and then proved that half-line CE is just the half-line we are looking for, i.e., the half-line that divides angle BCD just as we need it divided. Now let us try to prove that Euclid was correct.

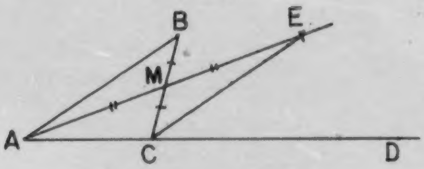

Figure 10

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What do you think? Which of the two angles BCD and ECD is equal to angle B? Evidently, angle BCE is equal to angle B.

10. True, but how can we prove it?

Then the teacher ends the discussion in the usual manner, i.e., examining triangles containing comparable angles, proving the angles equal, concluding with the equality of the angles BCE and B, which proves the original premise.

Example 2. Topic: "In any triangle the angle with greatest measure lies opposite the side with greatest length."

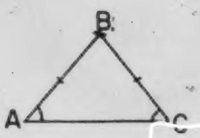

Figure 11

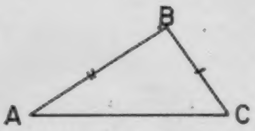

Figure 12

1. Let us recall the theorem of the isosceles triangle. In an isosceles triangle (Figure 11) the base angles are equal.

2. Correct. But the angles at the base are angles lying opposite equal sides of the triangle. Thus the formula must be stated differently: "In any triangle, equal angles lie opposite equal sides."

3. Now let us see what can be said about the angles lying opposite unequal sides of a triangle. Let us take triangle ABC, whose side AB is greater than side BC (Figure 12). So that we can use the theorem on the triangle with equal sides, let us try to apply this case to the case of a triangle with equal sides. How can we do this?

On the greater side BA we mark off from vertex B a segment BD equal to segment BC (Figure 13), then join points D and C.

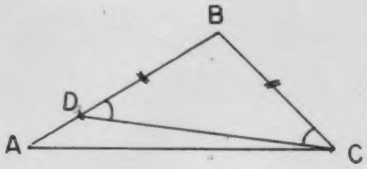

Figure 13

4. Correct. Then we obtain what isosceles triangle?

Triangle BDC with equal sides BD and BC and, therefore, with equal angles BDC and BCD.

5. And what angles must we compare?

Angle BCA and angle A.

6. But to compare them, we introduced auxiliary equal angles BDC and BCD. We shall compare angles BCA and A with these angles. $\angle BCA > \angle BCD$, since the second angle is included in the first; $\angle BDC > \angle A$, by the theorem on the external angle of a triangle.

7. Can we draw a conclusion about angles BCA and A from these two inequalities?

Yes, since angles BCD and BDC are equal and one of them may be substituted for the other.

8. What conclusion can we draw, using this substitution?

$\angle BCA > \angle BCD > \angle A$, and therefore, $\angle BCA > \angle A$.

Example 3. Topic: "Quadratic Equations."

1. Who can write an elementary equation having a root of 2?

Answer: $x - 2 = 0$.

2. Who can write an elementary equation having roots 2 and 5?

Answer: $(x-2)(x-5) = 0$.

3. Correct. But if we do multiply out the left-hand side, this equation becomes $x^2 - 7x + 10 = 0$, which has the same roots. If you were given the latter equation, how could you prove that it has the roots 2 and 5?

Answer: You factor it.

4. And how would you factor the trinomial $x^2 - 7x + 10$?

We represent its middle term as a sum (we decompose the middle term into two summands, we "splinter" the middle term) and transform the trinomial into a quadrinomial, $x^2 - 2x - 5x + 10$, in which we can combine like terms.

5. Correct. But can we factor any trinomial of this type in this way? For example, how do you factor the trinomial $x^2 - x - 12$; $16x^2 - 16x + 3$; $x^2 - 2x + 5$?

Answer: $x^2 - x - 12 = x^2 - 4x + 3x - 12 = x(x-4) + 3(x-4) = (x-4)(x+3)$; $16x^2 - 16x + 3 = 16x^2 - 4x - 12x + 3 = 4x(4x-1) - 3(4x-1) = (4x-1)(4x-3)$; we do not know how to factor the trinomial $x^2 - 2x + 5$—can it be factored?

6. You factored the first two trinomials correctly, and now you can probably solve these equations: $x^2 - x - 12 = 0$; $16x^2 - 16x + 3 = 0$.

   The first equation has roots -3 and 4, and the second, $\frac{1}{4}$ and $\frac{3}{4}$.

7. Correct. For the third trinomial, it is understandable that you were unable to factor it—it really has no factors. Not all of you, however, were able to find the factors for the first two trinomials, and even those who could had some trouble. The question arises whether we can find some general method for solving the problem whether a trinomial of the type $ax^2 + bx + c$ can be factored, and, if so, how.

Then the teacher familiarizes the pupils with the method of singling out a perfect square from a trinomial of the second degree such as $ax^2 + bx + c$; here he of course begins with particular cases and gradually approaches the most general case.

Example 4. Topic: "The theorem on the bisector of the internal angle of a triangle."

As a preliminary, for homework the pupils are to prove that, if in triangle ABC the sides AB and BC are unequal and AB < BC, the bisector of angle B divides the opposite side AC of the triangle into unequal parts AD and DC, where AD < DC, i.e., the smaller part of the base (AD) belongs to the smaller side AB, and the larger part of the base (DC)—to the larger side. This theorem is the result of another theorem: "If in triangle ABC sides AB and BC are unequal and AB < BC, then the median BM, dropped from vertex B of the triangle, divides angle B into unequal parts such that $\angle ABM > \angle CBM$." The proof is conducted using an ordinary construction (for theorems dealing with the median): The median BM is extended for a distance equal to it to point D (MD), and point D is joined to point C.

After checking this homework assignment, the teacher may conduct the lesson with a system of questions:

1. If in triangle ABC sides AB and BC are equal, how does bisector BD divide side AC?

   Into equal parts AD and DC.

2. Now let sides AB and BC of triangle ABC be unequal, and let us suppose that AB < BC. Now how does the bisector divide side AC?

   Into unequal parts AD and DC, where AD < DC, by the theorem already proved.

3. Correct. But Euclid, in his Elements, gave a more precise answer to this question. He proved that a perfect relation appears: AD:DC = AB:BC. Now let us try to find the proof of this statement. Look carefully at the proportion that we must prove and at the positions of the relative segments, given in a drawing.

   Segments AD and DC are located on side AC of angle A, and segment AB is on side AB of this angle.

4. Can you see what auxiliary line we must draw to construct a segment that would be the missing term in the proportion AD : DC = AB : ? .

   From vertex C we must draw a line parallel to bisector BD intersecting the extension of AB at some point E. Then we shall be able to use the theorem on the proportionality of segments formed on the sides of an angle intersected by parallel lines.

5. Quite right. Come to the blackboard and make the construction.

Using this course, the class proves the theorem, with the teacher's guidance.

Example 5. Topic: "General Properties of the Exponential Function."

1. What property does an arithmetical product of two numbers have if the multiplier is a proper fraction?

   In this case the product is smaller than the multiplicand.

2. Right. Using this theorem, what conclusion can we draw about a power whose base is a proper fraction and whose exponent is a positive integer--for example, these powers:

$$\left(\frac{1}{2}\right)^3, \left(\frac{3}{4}\right)^5, 0.24^8, \text{etc.?}$$

   All these powers are proper fractions, since, for example,

$$\left(\frac{3}{4}\right)^5 = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4},$$

and from multiplying the proper fraction 3/4 by the proper fraction 3/4, we obtain a number less than 3/4, i.e., again a proper fraction.

3. Correct. That means we can state the following theorem "In raising any proper fraction to a positive integral power, we again obtain a proper fraction."

   Well, what can we say about the positive integral power of an improper fraction, such as $\left(\frac{4}{3}\right)^5$?

   This power is an improper fraction, for in multiplying the improper fraction $\frac{4}{3}$ by the improper fraction $\frac{4}{3}$ we obtain a number greater than $\frac{4}{3}$, i.e., again an improper fraction, and so forth.

4. It appears we can now formulate a general theorem: "In raising any positive fraction to a positive integral power, we obtain a fraction having the same form (type, character) as the base." By the form (type, character) of a fraction we mean here its property of being proper or improper.

5. Now it is completely natural to ask ourselves what conclusion can be made about a power if the base of it is a proper or improper fraction and the exponent is a negative integer, for example,

$$\left(\frac{1}{2}\right)^{-3}, \left(\frac{3}{4}\right)^{-5}, 0.24^{-8}.$$

To answer this question, the teacher asks the pupils to calculate the powers he has written and to draw a general conclusion from them. "In raising my positive fraction to a negative integral power, we obtain a fraction with an inverse form (type, character), i.e., an improper fraction if the base is a proper one, and a proper fraction if the base is an improper one."

Similar reasoning, stemming from the same property inherent in the product of two numbers when the multiplier is a proper or improper fraction, leads the class to establish a second general theorem. "If the base of a power is a proper (improper) positive fraction, with an increase of the exponent, the power decreases (increases)." Extension of the above theorems to cover positive and negative fractional exponents can also be done mainly through the pupils' own efforts.

These examples we have given may help the teacher to form a certain notion of the essence of the heuristic method. However, the system of questions that may be asked on each topic can be extremely varied, and quite a broad range still remains up to the teacher's creativity.

The Laboratory Form of Instruction. The form of instruction which is built on drawing senses (mainly sight, touch, and muscular effort) into active participation is called the laboratory method. The following types of activities may be included in the laboratory form of instruction.

Independent solution of examples and problems during class

This type of work is done either by the class as a whole (all pupils are given exactly the same problem to do) or by giving the pupils individual data (they are given variants of a problem, with the brighter children getting the more difficult variants). The teacher, watching the solution of the assignment, can get an idea of the level of knowledge of many pupils, their ability to apply their knowledge in practice, the firmness of the skills they have acquired, their initiative and aptitudes for an independent and possibly creative method of finding the most rational ways to solve a problem.

This type of work has not a little significance when we consider that the pupils' competition with one another gives rise to a healthy and beneficial mutual influence and that all who cope successfully with an assignment experience moral satisfaction when their efforts are verified.

Execution of graphic exercises

The following belong in this category: construction of diagrams and graphs of functions, graphic illustration of the solution of equations and inequalities and systems of equations and inequalities, graphic solution of equations and systems of equations and inequalities.

Constructing diagrams is done in the fifth and sixth grades for helping the pupils to understand the idea of change and gives rise to the idea of functional relationship. It induces the pupils to apply their efforts actively, to be creative. Joining in this group work gives many of the pupils real joy. Lessons devoted to constructing diagrams are very lively and at the same time very useful for mastering the concepts and ideas that make up the content of the lesson topic.

As early as the sixth grade, there is a transition from the construction of diagrams to the construction of graphs of functions. This prodigious task, which is organically connected with the problem of constructing a mathematics course on the idea of the function, is fulfilled gradually, in a definite sequence; it is examined with the greatest completeness and depth in the upper grades as a measure of compiling the appropriate material. There are two stages in its resolution.

First the pupils acquire skill in constructing graphs by points. This is a very important stage; construction of graphs by points is also applicable in the second stage and in general becomes an obligatory element in the process of graphing functions. In the first stage one acquires a fixed conception of the graph as a locus of points whose coordinates satisfy a given equation.

The pupils' activity in the first stage involves constructing the points of a graph by given coordinates, first on the blackboard covered with a coordinate net and later in an arithmetic notebook, containing squared paper. Eventually the pupils need more independence and initiative in their assignments on graphing a function. Here, of course, the work may be done either by the whole class or in individual variants. In the latter case all variants can require constructing a graph of the same function, but from various systems of points so that all points constructed by the individual pupils are visible not only in their notebooks but also on the blackboard, resulting in a curve constructed from all points found by the individual pupils being presented on the blackboard.[2]

To some extent in the seventh grade, but mostly beginning in eighth grade, the pupils construct graphs of functions after a preliminary analytic investigation of their course. This does not mean that more thoroughly covered properties of functions should not first be examined on a graph that has been constructed (from points and partly on the basis

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[2] Prof. V. L. Goncharov used such a device in an experimental check of the algebra textbook he wrote.

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of a preliminary analytic examination) and only then proved analytically. The construction of the graph of functions, based on an analytical investigation of their course, is of great value in combining the pupil's mental and physical activity and in joining theory and practice harmoniously.

These exercises usually interest the pupils very much, especially if the functions illustrate a law that governs a process studied in science (physics, chemistry, technology, biology), a natural phenomenon.

Exercises in constructing graphs of functions have such great value for mathematics study in the school and higher educational institutions and for the pupil's future practical work that the teacher should not limit himself to the types of functions indicated in the curriculum, but should feel free to introduce other functions that are structurally similar to those being studied. Such, for example, are the linear fractional function $y = \frac{ax + b}{cx + d}$, the trigonometric function $y = a \sin (bx + c)$ and others which are easily related to functions that are already familiar to the pupils.

The construction of graphs, for the purpose of illustrating the solution of equations and inequalities, systems of equations and inequalities, is in essence not only an illustration, but also the beginning of a contribution to this solution. Such a contribution is especially clearly shown in solving a system of two equations, of which one or both are of the second degree. Here it is important to establish, by constructing graphs, the number of points of their intersection and consequently the number of solutions of the system.

Finally, the construction of graphs has yet another valuable practical application. It is used for a graphic solution of equations and systems of equations. This means of solving equations may be successfully applied whenever analytic solution of an equation or system of equations involves difficulties.

Execution of measuring tasks

This category of laboratory work includes the measurement of lengths, areas, volumes, and angles (done in the classroom) and measurement in a locality, done partly in the classroom but mostly out of it. The pupils usually begin doing measurements of the first type in the lower grades when learning information about geometry that is relevant there. Gradually these exercises in measurement become more complex (for example, in field sketching and familiarization with elementary measuring instruments) and prepare for the transition to measurement in a locality.

Both types of exercises in measuring have, of course, a specific place in mathematics instruction, but their significance increases as polytechnical training is introduced into the school. The skills acquired in doing these exercises provide a valuable store of practical knowledge and abilities, which the Soviet school should be giving its students at this time.

The execution of modelmaking tasks

This type of laboratory work, which is not yet fully realized in the school, occupies an increasingly important place in mathematics teaching in connection with the recent introduction of polytechnical training. Although mathematics teachers, striving to develop visual conceptions in their pupils, used to use prepared models, modern methodology no longer considers it possible to use only the pupils' contemplative activity, but requires that they take part in creating a model, in constructing it, which would activate their thinking and deepen their knowledge in one area or another. There can hardly be any doubt of the correctness of this methodology. But modelmaking plays a considerable role in two other respects; it promotes development of the pupils' elementary skills in constructing the details of models and in handicraft (in working with wire, cardboard, glass, tinplate) and develops the pupils' creative aptitudes, forcing them to construct and create models by themselves, often of complex geometric figures illustrating the proof of theorems and the solution of problems. Such are the models for the theorem on the shortest distance between two intersecting straight lines or the solution of the problem on constructing a straight line through a given point and cutting across two given intersecting lines (here it would be best to have a model containing the given intersecting lines in parallel planes).

The construction of models illustrating the course of the change of some elements of a figure (flat or spatial) in terms of the change of some parameter opens a vast field for the pupils' creative activity. This idea of investigating change in the elements of a figure has lately been recognized clearly by many teachers, who apply it in their practice.

The Lecture Form of Instruction. The form that the teacher gives to the lesson depends on the content and the nature of the material to be studied. Whenever new material is to be given and conducting the lesson heuristically would present difficulties and not be expedient, the teacher resorts to presenting the material as a short lecture. This lesson form is used primarily in the upper grades, and its usefulness increases according to the proximity to completion of the school mathematics course; one must not think, however, that it is entirely excluded from mathematics teaching in the fifth through seventh grades. To be sure, the lecture form is the only way to present historical material, which, of course, is given even in the fifth grade in conjunction with the study of arithmetic.

It is completely necessary to accustom the children to "listening to lectures." Listening to lectures is paramount to studying some part of the material on the basis of related and sequential information given by the teacher. This trains the pupils' powers of application, forcing them to be "active listeners" (to manifest activity in what would seem an inactive manner, by simply listening), prepares them to listen to lectures in the higher educational institution, and develops the ability to listen and try to grasp the words of the lecturer, speaker, or interlocutor, an ability that is valuable for every educated person. The teacher's lecture exposition of some part of the material acquires a special significance in mathematics instruction too, because it gives the pupil a model of speech connected, that is sequential, logically constructed, and stylistically correct.

The reader who has carefully familiarized himself with the above examples of how to conduct a lesson in heuristic form has probably noticed that it is not always possible to do so completely, in "pure" form so to speak. Most often the teacher must interrupt the smooth alternation of questions and answers with short explanations or interjections, which have essentially the form of a lecture. Thus the lecture form of communicating the study material occurs inevitably. The teacher's experience should suggest to him how much the lecture form must be used in each individual case, and, as a rule, he should not permit this form whenever it is possible and more useful to use the heuristic form. We repeat, however, that in the ninth and tenth grades the lecture form is to be preferred to the heuristic whenever the latter does not evoke enough of the pupils' mental tension—whenever it does not awaken in them the desired interest in the problem but gives rise to boredom and a certain ironic attitude toward the attempts of the teacher who tries to make a mystery of material that is no longer a mystery for them, children who are "no longer babies."

Finally, the exposition of a particular part of the study material in a lecture is undoubtedly of use because it gives the teacher the right and the opportunity to demand of the pupils a corresponding exposition of that same material in their answers during the lesson. Moreover, he can make this material the subject of written presentation by all the pupils through which he can establish the extent to which his chosen form of exposition has achieved its aim.

There are often times when the teacher comes across a convenient chance to begin the exposition of a topic in class that could be developed and finished by the pupils in their club. For example, the theorem on the bisector of the internal and external angle of a triangle can be completed in the club by a lecture on the two harmonic points (the points dividing the given segment internally and externally in precisely the same respect) and on the Apollonian circle. The unit on dividing a segment in mean and extreme ratio can be expanded to the unit on the golden mean. The unit on regular polyhedra naturally entails a lecture on Euler's theorem about convex polyhedra.

So, too, the unit on isolated equations of higher powers, given in the eighth and tenth grades, leads to a lecture on equations of higher powers; the unit on the infinite decreasing geometric progression may be concluded with a report on converging and diverging numerical series. A lecture given in class on the essence of the method of mathematical induction is continued in the club as a report on the same topic.

These examples, whose number may be greatly increased, will clarify for a teacher how study material, presented in class primarily by lecture, can be a good basis and stimulus for a pupil's report in the mathematics club.

Methods of Mathematics Instruction

As is well known, analysis and synthesis, induction and deduction are methods of scientific investigation--methods of seeking the proof of the correctness of various scientific relations (truths, facts, properties). When scientific relations become the subject of instruction, the teacher, as far as possible, puts the pupils in the position of persons who must rediscover or establish these relations independently.[3] The teacher "organizes" the pupils' reasoning and directs it in a definite channel. To do this, he uses the same scientific methods--thus the methods of scientific investigation become methods of instruction. Finally, the path taken by instruction should necessarily coincide with that of scientific investigation; another path may be taken in instruction, reasoning may be organized differently, but still this reasoning will be realized through the same scientific methods-- analysis and synthesis, induction and deduction. This is easily seen in examples.

1. The fact that "in any triangle the greater angle lies opposite the greater side" was discovered inductively and then proved deductively; but we do not know how the first deductive proof was made. And yet the teacher should do this proof as naturally as possible and promote mastery of it; he has a powerful means at his disposal--analysis, whose application shows the most natural way to find this proof.

2. The fact that "the three bisectors of a triangle intersect at one and the same point" can also be discovered inductively; deductive proof is easily found too, through analysis. But although one may discover inductively that "the three altitudes of a triangle intersect at one and the same point," we cannot confirm that ordinary deductive proof of this fact, which can also be easily found through analysis, coincides with what someone first established. But, for our purposes of instruction, this has no significance; for us it is only important that analysis leads us to a sufficiently natural and visual method of proof of this statement.

The Analytic-Synthetic Method. This method is an organic combination of two methods--the analytic and the synthetic.

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[3] Unfortunately the teacher often has only a limited time for this.

Analysis

Analysis is of two types.

1. The first type consists in the following. To establish a relation Q, one tries to find a relation or system of relations A_1 such that Q comes as a result. Then for each of the relations of A_1 one finds that relation or system of relations A'_2, A''_2, A'''_2,... from which relation A_1 follows accordingly. This process continues until all relations are covered, either those given in the conditions of the statement being proved or those established earlier and thereby considered true.

To give complete clarity to this general description of the process of analysis, let us cite some examples.

Example 1. Theorem: In any rhombus the diagonals are perpendicular.

1. Quadrangle ABCD is a rhombus.
2. AC and BD are its diagonals.
3. AC X BD at point O.

AC $\perp$ BD?

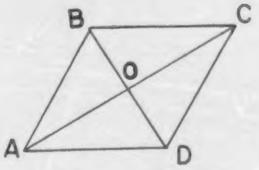


Figure 14

The following is an analysis of the proof of this theorem: (a) to prove that AC $\perp$ BD it is sufficient to prove that BO $\perp$ AC. i.e., that (b) BO is the altitude of triangle ABC. (c) For this, one need only establish that ABC is an isosceles triangle and that BO is its median, that is, that (d) AB = BC and (e) BO is the median of triangle ABC. Relation (d) follows from condition 1, and relation (e) follows from conditions 1, 2, and 3 of the theorem.

Let us look a little more closely at the links in this chain. Note that not only does relation (a) follow from (b), but relation (b) follows from relation (a); thus relations (a) and (b) are equivalent; they can fully replace each other. We can say that relations (a) and (b) are different in form but identical in content.

This is not valid for relation (b) and the system of relations (c), (d), and (e). Relation b follows from the system of relations (c), (d), and (e), but the system of relations (c), (d), and (e) does not follow from relation (b). Indeed, from the fact that BO is the altitude of triangle ABC it does not follow either that this triangle is isosceles or that BO is its median. For relation (b) to be valid, it is not necessary that the system of relations (c), (d), and (e) be valid, but it is sufficient. The presence of the system of relations entails the presence of relation (a), which is subject to proof.

Example 2. Theorem: A circle may be inscribed in any triangle.

ABC is a triangle
Can a circle be
inscribed in ΔABC?

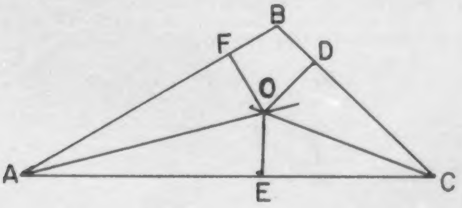


Figure 15

Analysis of the proof of this theorem may be presented in the following scheme.

(a) In triangle ABC a circle may be inscribed?
↑
(b) There exists a circle tangent to all sides of triangle ABC?
↑
(c) There exists a point equidistant from all sides of triangle ABC?
↑

(d) All points of the bisector of angle A are equidistant from sides AB and AC (theorem on the bisector of an angle).	All points of the bisector of angle C are equidistant from sides CB and CA (theorem on the bisector of an angle).	The bisectors of angles A and C intersect (property of non-parallel straight lines).

In this analysis we were looking for relation (b), from which relation (a) follows, then relation (c), from which relation (b) follows, and finally the system of relations (d), from which relation (c) follows. We turn our attention, however, to the fact that (conversely) relation (b) is a consequence of relation (a) and relation (c) is a consequence of relation (b). Thus (a), (b), and (c) are equivalent, and each of them may fully replace either of the others.

But the system of relations (d) is not a result of relation (b); it consists of nonequivalent relations. Here there is only a unilateral relationship. For relation (c), which is subjected to proof, to be valid, it is sufficient that the system of relations (d) be valid.

It can be seen from these examples that this type of analysis consists in seeking the logical path to be followed from the given relations to the relation being proved, i.e., it consists in establishing the path of proof that the given relations are conditions that are sufficient to realize the relation to be proved. This type of analysis is applied in *proving propositions*.

2. The second type of analysis, applied in solving problems (geometric or algebraic), consists in the following. *Beginning from the assumption* that the unknown figure or value of a quantity exists, we seek those relations that are the consequences of this assumption, then the relations that derive from these consequences, continuing until we come to the conclusion which may serve as the original relation in a chain of inverse propositions. Obviously we find in this manner the condition or system of conditions necessary for the existence of the unknown figure or value.

Thus, where the first type of analysis has the purpose of proving that known relations (given in the conditions of the proposition) are *sufficient* for the existence of the conclusion of the proposition, the second type of analysis helps establish the conditions *necessary* for the existence of a value or a system of values of an unknown quantity, so that after this (by synthesis) one may select those that are then sufficient, and, if required, add the new sufficient conditions.

Example 3. *Problem*: Construct a right triangle given its hypotenuse a and the radius r of a circle inscribed in it.

In the analysis of the solution of the problem, we assume that the problem is solved, i.e., that the unknown triangle is constructed (Figure 16). Let this be triangle ABC in which $AB = d$, $BC = a$, O is the center of the inscribed circle, $OD = OE = OF = r$.

Looking at the drawing, we see triangle BOC is determined by triangle ABC. In triangle BOC, $BC = a$, the altitude OD is equal to r, $\angle BOC = 180° - \frac{B + C}{2} = 180° - 45° = 135°$, so that the problem of constructing triangle ABC from elements A, a, r comes down to constructing triangle BOC from the angle at vertex BOC, from the base a, and the altitude r.

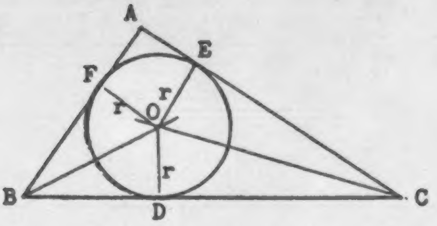


Figure 16

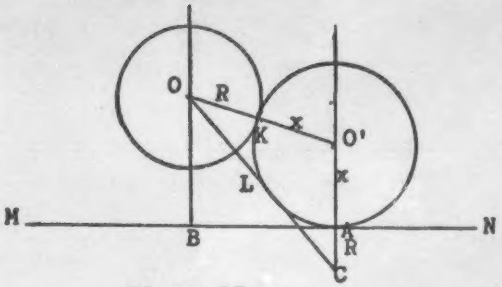


Figure 17

Example 4. Problem: Given a circle of radius R and line MN at a distance d from the center 0 of this circle, construct a circle tangent to circle 0 and line MN at a given point A on this straight line.

We analyze the solution of the problem. For this we assume that the problem is solved, i.e., that the unknown circle is constructed (Figure 17). Let this be a circle with center 0', tangent to circle 0 at point K and to line MN at the given point A.

It can be seen from the drawing that the center 0' of the unknown circle is on the perpendicular AC, drawn to line MN from point A. At first we can make no further conclusions about the location of the center 0', but, inspecting the drawing carefully, we notice that the center 0' is at a distance from 0 equal to the sum of radius R of the given circle and radius x of the unknown circle; from point A it is at a distance x. This leads us to think that if we mark off segment AC equal to radius R on the extension of the perpendicular AC, the center 0' is equidistant from points 0 and C, and, consequently, will be on the perpendicular to LO', constructed to segment OC from its center L. We conclude that the figure we assumed constructed determines the center 0' of the unknown circle as a point:

(1) located on the perpendicular AC erected from point A to MN;

(2) equidistant from points 0 and C and therefore on the perpendicular LO', constructed to segment OC from its center L;

(3) which is the point of intersection of the perpendiculars AC and LO'.

From this we conclude that to construct the center of the unknown circle, we must construct perpendicular AC and perpendicular LO'.

Examples 3 and 4 permit us to make the general conclusion that analysis in solving problems on the construction of some figure F from its given properties α consists in finding a figure or system of figures F' with properties α' such that it is wholly determined by figure F, so that in order to construct figure F it is necessary that a figure or system of figures F' be constructed.

Example 5. Problem: A pool has two pumps. The first pumps water into the pool. The second can pump out an equal amount of water, but it takes h hours longer. When both pumps are working together, the pool is filled in a hours. How long does it take the first pump to fill the pool?

To solve this problem, we designate the unknown number of hours by x and compose an equation,

$$\frac{a}{x} - \frac{a}{x+h} = 1. \qquad (1)$$

How did we arrive at this equation? We first assumed that when $h > 0$ and $a > 0$, then we can assert by considering the quantities h and a: There exists some value x of the unknown quantity (the time the first pump takes to fill the pool) that is the answer to the problem. By reasoning we then established that this value of the unknown quantity satisfies equation (1). Thus it turned out that for the statement:

(a) there exists a value x of the unknown quantity such that it answers the problem, entailing the statement that

(b) this value x satisfies equation (1).

Consequently, the fulfillment of statement (b) is a condition necessary for statement (a) to be valid. In other words, only that value of the unknown quantity which is a root of equation (1) can answer the problem.

We cannot, however, draw the inverse conclusion. From the fact alone that some number is the root of equation (1) it does not follow that it answers the question of the problem. In other words, fulfillment of statement(b)is not a condition sufficient for statement(a)to be valid.

Therefore, having solved equation (1), we shall have to be additionally convinced that one or another of its roots gives the answer to the question of the problem, or—if none of the roots of equation (1) satisfies this requirement-- that the problem has no solution at all.

But how can we be convinced of this? Having carefully considered the idea of the unknown value x whose existence we have assumed, we shall try to establish those supplementary relations satisfying it. It is easily understandable that the unknown value x satisfies not only equation (1), but also the relation

$$0 < x < a, \qquad (2)$$

which in view of this expresses still another condition that is *necessary* for the existence of a value x of the unknown quantity, which would answer the problem.

But it is not hard to see that relations (1) and (2) are not independent. If $x > 0$, from equation (1) given in the form

$$\frac{a}{x} = \frac{a}{x+h} + 1$$

it follows that $\frac{a}{x} > 1$, i.e., that $x < a$. There are two mutually independent conditions:

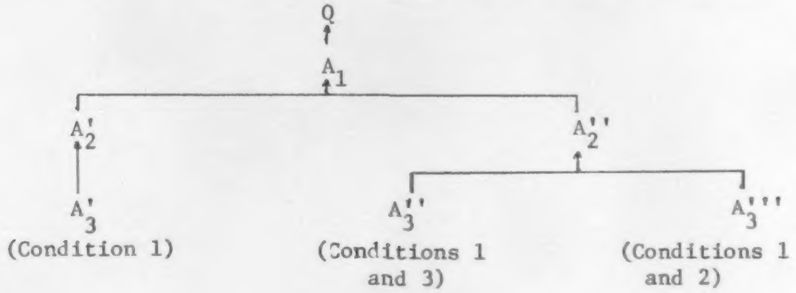
$$\frac{a}{x} - \frac{a}{x+h} = 1, \text{ and } x > 0,$$

which are *necessary* for there to exist a value of an unknown quantity which would answer the problem. Neither of these conditions in itself is sufficient. One may pose the question of the sufficiency of the system (combination) of these mutually independent conditions. We shall show below that this system is indeed sufficient for there to exist a value of the unknown quantity that will answer the problem.

Synthesis

To each of the two types of analysis there corresponds its own type of synthesis. Let us examine them.

1. Assume that, in trying to find a proof of some proposition Q, we have made an analysis, as represented by this scheme:



Let A_3', A_3'', and A_3''' be those statements that are direct consequences of the data of the conditions of the theorem, as indicated (in parentheses). Then it is clearly possible to form an argument consisting of a converse transition:

(a) from condition 1 of the theorem--to statement A_3' and from it to statement A_2';

(b) from conditions 1 and 3--to statement A_3'';

(c) from conditions 1 and 2--to statement A_3''';

(d) from statements A_3'' and A_3'''—to statement A_2'';

(e) from statements A_2' and A_2''--to statement A_1;

(f) from statement A_1--to proposition Q.

This method of establishing some relation, consisting in a sequential transition (with the aid of logical inferences) from the given conditions of a theorem to be proved to its conclusion, is the first type of synthesis.

Let us return to examples 1 and 2.

Example 1. Synthesis consists in the following (page 55):

(a) from condition 1 we find that AB = BC, and from conditions 1, 2, and 3 we conclude that BO is the median of triangle ABC;

(b) hence we infer that BO is the altitude of triangle ABC, i.e., that $BO \perp AC$ and, consequently, $BD \perp AC$.

Example 2. Synthesis consists in the following (page 56):

(a) since the bisector of any angle is a locus of points equidistant from its sides, the bisector of angle A contains all and only those points equidistant from sides AB and AC, and the bisector of angle C contains all and only those points equidistant from sides CB and CA; these bisectors (from the property of nonparallel lines) intersect at some point 0;

(b) therefore point 0 is equidistant from all sides of triangle ABC, i.e. if $OD \perp BC$, $OE \perp AC$, $OF \perp AB$, then $OD = OE = OF$;

(c) drawing a circle of radius OD with its center at 0, we obtain a circle tangent to all sides of triangle ABC (by the theorem of the straight line drawn perpendicular to the radius at the end of the radius lying on the circle), i.e., an inscribed circle.

2. The second type of synthesis consists in establishing that the conditions found necessary for the existence of the unknown figure or the values of the unknown quantity are also sufficient.

Examples 3 and 4. In construction problems, synthesis is made after the unknown figure is constructed and is nothing but a proof that the conditions, found through analysis and used in the construction, necessary for the existence of the unknown figure are also sufficient.

In solving example 3 it was established that in order to construct the unknown triange ABC it was necessary to construct triangle BOC. But this condition is at the same time sufficient. Indeed, having constructed triangle BOC from its base a, the altitude r, and the angle (equal to 135°) at vertex 0, and having drawn half-lines BA and CA, forming angles with half-lines BO and CO respectively, these angles are equal to angles OBC and OCB and intersect at some point A, and we obtain the desired triangle. Indeed:

(a) half-lines BA and CA intersect, and at right angles, since $\angle OBC + \angle OCB = 45°$, and $\angle ABC + \angle ACB = 2(\angle OBC + \angle OCB) = 90°$;

(b) 0 is the point of intersection of the bisectors BO and CO of angles B and C of triangle ABC, i.e., the center of the circle inscribed in it.

(c) altitude OD of triangle BOC is the radius of this inscribed circle in triangle ABC, but this altitude is exactly equal to r.

In the solution to example 4 it was established that in order to construct the center 0' of the unknown circle one must construct:

(a) perpendicular AC, drawn from the given point A to the given straight line MN;

(b) perpendicular LO', drawn to segment OC from its center L.

But this condition is simultaneously sufficient. Indeed:

(a) straight lines AC and LO', perpendicular to the sides of the angle, intersect at some point 0';

(b) from the equality $0'C = 0'0$ presented in the form $0'A + R = 0'K + R$ it follows that $0'A = 0'K$;

(c) in view of this, the circle circumscribed about the center 0' with radius 0'A passes through point K, so that point K is the common point of circles 0 and 0', lying on their line of centers 00' and, consequently, the point of their tangency;

(d) the straight line MN, passing through the end-point A of the radius 0A and perpendicular to it, is tangent to the circle constructed with its center at 0'.

These proofs that the conditions found through analysis, necessary for constructing the unknown figures in problems 3 and 4, are at the same time sufficient conditions for it; they are the stage of the solution of these problems, which is called synthesis.

Example 5. Analyzing this problem, we arrived at the conclusion that in order for a value x of the unknown quantity to exist it must satisfy a system of relations:

$$\frac{a}{x} - \frac{a}{x + h} = 1, \; x > 0. \qquad (3)$$

Having solved the equation constituting the first necessary condition, we can replace this sytem with an equation equivalent to it:

$$x = \frac{\sqrt{(h^2 + 4ah)} - h}{2}. \qquad (4)$$

This value of the unknown quantity may be viewed as the answer to the problem, for it is the value of that interval of time during which the first pump fills the pool. Indeed, it follows from the equation that the part of the pool filled by the first pump in 1 hour exceeds the part of the pool emptied by the second pump in 1 hour by exactly the part of the pool filled after both pumps have been operating together for 1 hour.

The above proof that system (3) of the conditions necessary for the existence of a value of the unknown quantity is at the same time sufficient, for this constitutes that stage of solution of the problem, with the help of the composition of an equation expressing its condition, which is called synthesis.

3. The analytic-synthetic method is successfully applied whenever a theorem may be presented in the form of a problem.

Example 6. To prove in class the theorem on the square of a side opposite an acute angle, the teacher does not tell the pupils the formula but asks them to solve this problem: "Given lengths a and b of two sides of a triangle containing an acute angle, find the length of the third side."

In a preliminary discussion with the pupils, the teacher establishes that, since a triangle is determined by three independent elements, knowledge of lengths a and b is not enough to solve the problem and that another element must be known; what kind of an element this is, is explained in the process of solving the problem. Then the class does the following analysis under the teacher's guidance (see Figure 18, where BC = a, AC = b, AD $\perp$ BC):

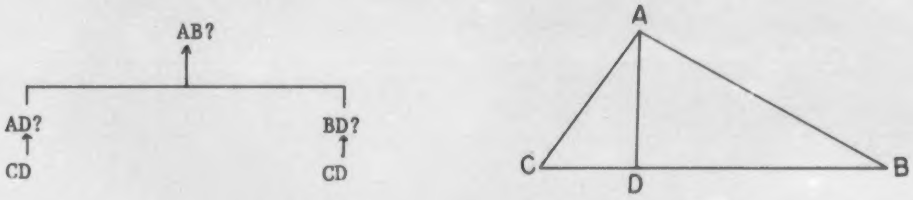


Figure 18

This analysis shows that this third element, knowledge of which is *necessary* for solving the problem, is segment CD, i.e., the projection of one of the two given sides (CA) onto the other given side (CB).

Designating the length of this projection CD by b_a (to show that CD is the projection of side b onto side a), we synthesize:

1) $AD^2 = b^2 - b_a^2$;
2) $BD = a - b_a$;
3) $AB^2 = AD^2 + BD^2 = b^2 - b_a^2 + a^2 - 2ab_a + b_a^2 = a^2 + b^2 - 2ab_a$.

This order may be applied whenever the conclusion of some definite formula is replaced by calculation of that magnitude designated by the left part of a formula and whose value is expressed by the right part--for example, geometric formulas for calculating the nonfundamental elements of a triangle, or trigonometric formulas, known as formulas of addition and subtraction, and their consequences.

4. Let us summarize and state one general consideration.[4]

For clarity and brevity we shall introduce the following designation. We shall write symbolically a theorem that consists of a relation β following from relation α in this manner: $\frac{\alpha}{\beta}$. This notation will signify that "relation α is given" and that "relation β must be proved."

Let us assume that

(1) each of these theorems is true:

$$\frac{\alpha}{\beta}, \frac{\alpha}{\gamma}, \frac{\alpha}{\delta}; \quad (A)$$

---

[4] Point 4 is devoted to a deeper development of the logical aspect of the question and is designed to familiarize the teacher with it.

(2) relations β, γ, δ are independent, i.e., no one of them nor any group of them entails any of the others;

(3) if some relation ε is the consequence of neither any relations β, γ, or δ nor any group of them, then theorem $\frac{\alpha}{\varepsilon}$ is untrue.

In this case we may confirm that the theorem is true, i.e., that relation α follows from the combination of relations β, γ, δ.

We may formulate this theorem in words thus:

Theorem. If for the existence of relation α it is *necessary* that *each* of the mutually independent relations β, γ, δ, and *only* these relations, exist, then for the existence of relation α it is *sufficient* that a *system* of relations β, γ, δ exist.

Proof (indirect method). Assume that relation α does not exist, despite the existence of relations β, γ, δ subject to conditions 1-3.

Let us examine under what conditions this is possible. From condition 1 it follows that the nonexistence of each of the relations β, γ, δ entails the nonexistence of α, so that the reason for the nonexistence of relation α could be the nonexistence of any one of the relations β, γ, δ. But, according to the conditions, this does not occur. Therefore, the invalidity of the relation α cannot be due to the invalidity of any one of the relations β, γ, δ. We still must find out whether the invalidity of relation α is the consequence of the invalidity of some relation ε independent of relations β, γ, δ. That cannot be, for then the theorem $\frac{\alpha}{\varepsilon}$ would be true, contradicting condition 3.

Thus the assumption that in the presence of relations β, γ, δ, subject to conditions 1-3, relation α does not occur must be rejected, i.e., relation α really is a consequence of relations β, γ, δ.

Example 7. For a number a to be divisible by 12 it is necessary that a be divisible by 3 and 4. In addition, the relations "a is divisible by 3" and "a is divisible by 4" are mutually independent. The relation "a is divisible by 2" is a consequence of the relation "a is divisible by 4"; the relation "a is divisible by 6" is a consequence of the system of relations "a is divisible by 2" and "a is divisible by 3." There is no relation distinct from the two indicated independent relations, and the two that are the consequences of these independent relations can be the consequence of the relation "a is divisible by 12," for the number

12 has, besides the trivial divisors 1 and 12, *only* the divisors 3, 4, 2, 6. These considerations permit us to conclude that if *a* is divisible by 3 and 4, it is divisible by 12. Indeed, the indivisibility of *a* by 12 could be the consequence only of its indivisibility by either 3 or 4, and this contradicts the conditions.

The converse theorem, having the following content, is also true.

If

(1) each of the theorems $\frac{\alpha}{\beta}$, $\frac{\alpha}{\gamma}$, $\frac{\alpha}{\delta}$; (A) is true,

(2) relations β, γ, δ are independent, and

(3) theorem $\frac{\beta, \gamma, \delta}{\alpha}$ is true,

Then we may confirm that if relation ε is not a consequence of any one of the relations β, γ, δ or any group of them, then theorem $\frac{\alpha}{\varepsilon}$ is untrue, i.e., the relation ε is not a consequence of relation α either.

We may formulate this theorem in words:

Converse theorem. If for the existence of relation α it is *necessary* that *each* of the mutually independent relations β, γ, δ exist, and it is *sufficient* that the *system* of these relations exist, then there is no single relation ε, independent of relations β, γ, δ, which would be necessary for the existence of relation α.

Proof (indirect method). If there were some relation ε whose existence would be necessary for the existence of relation α, then it would appear that two theorems would exist simultaneously:

$$\frac{\beta, \gamma, \delta}{\alpha} \text{ and } \frac{\alpha}{\varepsilon},$$

i.e., that theorem $\frac{\beta, \gamma, \delta}{\varepsilon}$ would be true, which contradicts the conditions. Therefore, proposition $\frac{\alpha}{\varepsilon}$ must be rejected.

Example 8. For a number a to be divisible by 12 it is necessary for each of two independent relations, "a is divisible by 3" and "a is divisible by 4," to be fulfilled, and it is sufficient that the system of these relations be fulfilled. No relation ε that would not depend on the two indicated relations and whose fulfillment would be necessary for the divisibility of a by 12 exists, for from the theorems

$$\frac{\text{a is divisible by 3, a is divisible by 4}}{\text{a is divisible by 12}} \text{ and } \frac{\text{a is divisible by 12}}{\varepsilon}$$

there would follow the theorem $\frac{\text{a is divisible by 3, a is divisible by 4}}{\varepsilon}$, i.e., relation ε would depend on the system of relations "a is divisible by 3" and "a is divisible by 4," which contradicts the conditions.

These examples of (direct and converse) theorems, expressing the relation between conditions necessary and sufficient for the existence of a relation, show why we attempt through analysis to establish all mutually independent necessary conditions in order to find a system of sufficient conditions.

In examining example 4, we established that, for the existence of the unknown circle, it is necessary that there exist:

(1) a perpendicular AC, drawn from point A to line MN;

(2) a perpendicular LO', drawn from the center L of segment OC;

(3) a point of intersection O' of these perpendiculars.

Condition 3 is obviously a consequence of conditions 1 and 2 (perpendiculars drawn from the sides of an angle intersect), i.e., it is dependent on them. But conditions 1 and 2 are independent. Therefore the system of conditions 1 and 2 is sufficient for the unknown circle to exist.

The Inductive and Deductive Methods of Instruction. In this section we shall examine the application of the inductive and deductive methods of mathematics instruction.

Incomplete induction

1. The inductive method of mathematics instruction is the method of establishing some facts (truths, propositions) and consists in finding general conclusions on the basis of a definite number of separate (particular) observations. Thus, instruction conducted inductively is built on material obtained from direct perception of actual objects and facts. It is fully understood that this method alone can and should be applied in the beginning stages of mathematics instruction, but it also retains its value at other levels of this instruction when the pupils show their need for establishing general attitudes by the method of abstraction as well as sufficient aptitude for it.

(a) Having established the criteria for divisibility by 2 and 5, 4 and 25, 8 and 125, the teacher turns to seeking the criteria for divisibility by 3. He begins by examining individual numerical examples and observes in each of them that if a number is divisible by 3, the sum of its digits is divisible by 3, and, conversely, if the sum of the digits of a number is divisible by 3, the number itself is divisible by 3. After the pupils have been convinced of these facts through many examples, they will hardly be inclined to doubt them, but of course they will want to know the "reason" behind these phenomena. This is a good chance for the teacher to ask them to solve this problem by themselves, as they had found the criteria for divisibility by 2, 4, and 8.

(b) The inference of the formula for solving the equation of a perfect square, based on isolating a perfect square from its left part, is always begun in this way, by solving diverse numerical examples of gradually increasing complexity. In these examples the pupils learn the general principle that, for the left-hand side of each quadratic equation, one may establish precisely whether it may be decomposed into linear factors and, if so, which ones.

(c) Familiarizing sixth-graders with the concept of the altitude of a triangle, the teacher draws on the blackboard some scalene triangles of various shapes and draws the altitude to the base in each of them. By examining these triangles the pupils "arrive at the conclusion" that if the base angles of a triangle are acute, the altitude falls on this base, and if one of the two base angles of a triangle is obtuse, the altitude falls on the extension of this base.

This conclusion, inferred purely inductively by the pupils on the basis of a series of drawings, seems fully convincing to them, completely believable; at this stage of their mathematical development it has not yet occurred to them that they may doubt an observed fact, and the teacher would be in error if he aroused their doubts now about the authenticity of the conclusions they have drawn. But when it is possible for the teacher to substantiate this fact, rigorously, he errs again if he does not take advantage of this opportunity.

Let us turn to the generality contained in the above examples a-c and constituting the essence of the concept of incomplete induction. In example 1 we were talking of the set of natural numbers whose digits add up to a number divisible by 3. It appeared that all the elements we investigated in this set have one and the same property--divisibility by 3. The question arose whether all elements of the infinite set being studied have this property; the pupils said yes.

In example 2 we looked at the set of all quadratic equations with numerical coefficients. It appeared that each equation we investigated had one and the same property. Concerning it, one could determine precisely whether its left-hand term could be decomposed into linear factors and which ones, or whether it could not be factored. One must determine whether all elements of the *infinite* set we examined have this property. This question is thoroughly solved by applying those same arguments to an equation in general form.

In example 3 we studied the set of triangles whose base angles are acute and the set of triangles having one of these angles obtuse. It appeared that all triangles of the first set have one and the same property--the altitude of each of them falls on its base. The triangles of the second set have another common property--the altitude of each of them falls on the extension of its base. As a result, the question arose whether one could confirm that *all* elements of the first *infinite* set and *all* elements of the second *infinite* set have such properties.

Thus the inductive method of instruction may be applied when it is necessary to establish that *all* elements of some *infinite* set M have one and the same (common) property α and when strict proof of this fact cannot be given to the pupils, or when the teacher nevertheless considers it necessary to first "discover" the fact with the help of inductive argumentation.

In examples 1 and 2 a strict proof may be given immediately after the fact is "discovered," and in example 3, only later, after all the necessary information has been accumulated. The induction used in examples 1-3 is called *incomplete* induction. We gave these examples only to illustrate the inductive method of instruction. They have no peculiarities which would separate them from the great number of other cases in which the teacher applies the inductive method.

We shall state a much more general idea. Application of the inductive method underlies the application of the so-called *concrete inductive method*, which constitutes one of the most valuable achievements of the methodology of teaching of mathematics, one of its most fruitful ideas. The concrete inductive method is contrasted to the abstract deductive method, which--according to teachers who are disposed toward it--should only gradually acquire a leading position in mathematics instruction in the upper grades, without displacing the concrete inductive methods, but coexisting with it. As its name indicates, the concrete inductive method requires that the entire concept or general principle (which in the final analysis must be invested in an abstract form) begins to be built up or created in the pupil's mind through a study of completely concrete images and propositions on the basis of an examination of a series of concrete examples that disclose (with sufficient definiteness, clarity, and completeness) the essence of the concept and basic idea of that proposition which is to be concluded. It is fully understood that this goal may be attained only by following the inductive method of the investigation and finding properties of concrete objects. In addition, by concrete examples we must of course mean not only objects and phenomena of the external world directly perceived by our senses, but all individual representatives of those concepts that are to be generalized.

Thus, whereas in example 3 each triangle drawn on the blackboard is directly perceived by sight, in example 1 each natural number whose digits add up to a number divisible by 3 is only an individual representative of the general concept of the natural number having this property. In this sense, each separate number whose digits add up to a number divisible by 3 is related to concrete objects; in the same sense, we call concrete each quadratic equation with definite numerical coefficients (example 2).

Complete induction

2. Sometimes that method is called inductive--true, not consistently enough--which consists of establishing some fact (principle, proposition, truth) by examining all singularly possible (mutually exclusive) cases and the proof of the justification of this fact in each of the possible cases.

(d) For a proof of the theorem, "An inscribed angle is measured by a half of the arc on which it is based," we establish its correctness for the only three possible cases; then the theorem may be considered completely proved.

(e) To establish the general rule for adducing the trigonometric function of any angle belonging to the segment (0° ... 360°) to a trigonometric function of an angle of the first quadrant, we infer this rule for each of the following only possible representatives of the value of an angle, given here as an algebraic sum: $90° \pm \alpha$; $180° \pm \alpha$; $270° \pm \alpha$; $360° \pm \alpha$, where $0° < \alpha < 90°$; after this the rule may be considered established generally.

Examples 4 and 5 share the fact that they illustrate the application of one and the same method. However, they are not the same. While the proof of the theorem of the inscribed angle cannot be presented without examining the definite sequence of the three cases studied, the conclusion of the rule for adducing trigonometric functions can be drawn immediately by using this common method. If the teacher desired to communicate this general method to the pupils, a preliminary examination of particular cases would be devoted to presenting the whole topic with a very useful concreteness and visuality. In this case, the device used by the teacher could be called inductive. But if the teacher decided to examine all only possible (and mutually exclusive) cases, here there would be no element of transition from particular observations to a general conclusion, but a general conclusion would be created by proving a theorem for all possible cases without exception, and the omission of even one case would make the proof fallacious, since there would exist that logical error which is called the incompleteness of a proof. The device used in examples d and e is called complete induction.

It goes without saying that the teacher is not obliged to consider all possible cases himself. He is helped by the pupils, who in individual cases, independently apply a course of reasoning indicated by the teacher. It is important, however, that the pupils know that to regard a rule as established solely on the basis of its being established in some, but not all, cases would be an error in logic.

Deduction

3. Let us turn to an examination of the deductive method of mathematics instruction. It consists in applying deductive reasoning to establish specific theoretical facts (truths, propositions). This reasoning consists of a system (chain) of sequentially realized syllogisms. As is known, in deductive reasoning we usually use enthymemes (syllogisms which lack one of their premises), but the teacher should nevertheless present deductive reasoning in its complete form through examples, i.e., so that all its syllogisms would be clearly separated.

Example. The proof of the theorem, "In any triangle the greater angle lies opposite the greater side" may be decomposed into the following syllogisms (Figure 19):

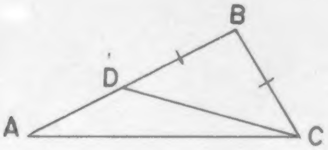


Figure 19

1. In any isosceles triangle the base angles are equal. DBC is an isosceles triangle with base DC. Therefore, $\angle BDC = \angle BCD$.
2. The sum of the two angles is greater than either angle; $\angle ACB = \angle ACD + \angle BCD$. Therefore, $\angle ACB > \angle BCD$.
3. If one of two parts of an inequality is replaced by a quantity equal to it, we obtain an inequality of the same sense as the given inequality; $\angle BCD = \angle BDC$. Therefore, $\angle ACB > \angle BDC$.
4. Every external angle of a triangle is greater than each of the internal angles not adjacent to this external angle. $\angle BDC$ is the external angle of triangle ADC. Therefore, $\angle BDC > \angle A$.
5. The inequality of angles is a transitive relation. $\angle ACB > \angle BDC$; $\angle BDC > \angle A$. Therefore, $\angle ACB > \angle A$, that is, $\angle C > \angle A$.

Even in this example it is clear that in a deductive proof of a mathematical statement one must use syllogisms of various modes. It is quite advisable, however, to dwell on the present mode having the most widespread usage, and examine it more deeply, showing what its essence is. Any syllogism of this type can be presented in this general form:

> Every element of class K has property α.
> Object A is an element of class K.
> Therefore, object A has property α.

Thus, syllogism 1 given above may be given the following form:

> Every triangle belonging to the class of isosceles triangles has the property that its base angles are equal.
>
> Triangle BDC belongs to this class (is an element of this class).
>
> Therefore, triangle BDC has the property that its base angles are equal.

Thus the inference we have examined consists in the fact that if we include some object A in class K according to some specific criteria (we relate it to class K), we ascribe to object A all those properties that determine class K. It is easy to understand that the major and minor premises of a syllogism together constitute just this base on which we construct a corresponding point of the proof. Thus, in the example cited for proving that $\angle BDC = \angle BCD$, we refer briefly to the theorem of the isosceles triangle, saying, "according to the theorem of the isosceles triangle"; but unfolding this brief reference, we see that it leads to the first two propositions (the major and minor premise) of syllogism 1.

4. As is known, the deductive method of reasoning has an especially wide application in proving geometric propositions (theorems). The teacher, however, should clearly recognize that we use chiefly this same method in proofs of propositions of arithmetic, algebra, and trigonometry too. The basic fault in teaching algebra, especially in grades 8-10, is that the study of its theory is not given the same significance as the study of theory in geometry. The theory of algebra is presented on a lower level and the demands on the pupils are of a lesser degree and scope. More and more often methodologists and teachers are indicating this serious fault as the origin of insufficient knowledge of algebra. The propositions constituting the theory of the school algebra course must be given a precise form that would isolate the conditions and conclusion of a theorem and allow the proof to be conducted strictly deductively to the same extent as is done in geometry. Meanwhile, besides the propositions clearly formulated as theorems (the theorem of Vieta, the theorem of Bézout), the school algebra course contains a great many

propositions having a completely precise content which are subjected to strict logical proof.

We cite several examples:

1. If the determinant of a system of two linear equations with two unknowns is distinct from zero, the system has one and only one solution. If this determinant is equal to zero, but at least one of the supplementary determinants of the system is distinct from zero, the system has no solution. If the determinant of a system and both its supplementary determinants are equal to zero, but at least one of the coefficients in the system of equations is distinct from zero, the system has an infinite set of mutually dependent solutions.

2. If the discriminant of a quadratic equation is a positive number, the equation has two unequal real roots. If this discriminant is equal to zero, the equation has two equal real roots. If the discriminant of a quadratic equation is a negative number, the equation has no real roots.

3. If we multiply the numerator and denominator of an algebraic fraction by any expression not reducing to zero, we obtain a fraction identically equal to the first.

4. If to both parts of an equation we add any number or any expression that does not lose its numerical sense in the system of values of unknowns constituting the solution to the given equation, we obtain a new equation equivalent to the given one.

5. There is no integer or fraction whose square would be equal to 2.

6. If $a > 0$, and m and n are natural numbers, then $(a^m)^n = a^{mn}$.

7. If $a > 0$, then when $a \gtrless 1$ and $\alpha > 0$, the inequality $a^\alpha \gtrless 1$ is correspondingly true, and when $a \gtrless 1$ and $\alpha < 0$, the inequality $a^\alpha \lessgtr 1$ is correspondingly true.

8. If $a > 0$, $b > 0$, $c > 0$, then $\log (abc) = \log a + \log b + \log c$.

We have cited some examples of algebraic propositions precisely formulated. Each of them represents a theorem that may and should (in the appropriate grade) be rigorously proved. But these examples are not exceptions. In algebra theory there are a great many theorems, and, in formulating them, the teacher is obliged to precisely isolate their conditions and conclusions (see examples 1-8) and to appeal clearly to each of the conditions in proving these theorems. All demands made on the proof of geometric theorems should be made on the proof of algebraic theorems. Everything said above about algebra is of course true for arithmetic and trigonometry, too.

As early as the fifth grade, when the teacher tells the pupils about the criteria for divisibility of natural numbers, he is afforded the opportunity to invest them in the if-then form of conditional propositions. He should strive to do this later too, during the entire period of arithmetic study, and then in other sections of mathematics.

Not every pupil can replace the short formulation of the proposition "the common multiple of two relatively prime numbers is their product" by its full formulation: "if two numbers are relatively prime, their common multiple is the product of these numbers." But just how important this is for developing full precision and clarity of reasoning the teacher will see when he has to conduct more complex deductive arguments with the pupils: A proof based imprecisely on all the given conditions and only these conditions ceases to be a proof.

Even more so than in algebra, it is possible to carry out strict deductive proofs in trigonometry as taught in the upper (ninth and tenth) grades. Here the pupils' general and mathematical development is quite sufficient for the teaching to be conducted on the necessary scientific level. Therefore, formulations of theorems in trigonometry and their proof should satisfy the demands made for scientific proofs; in particular, deductive proofs should be conducted with precise observance of references to all the conditions and all propositions used--as major and minor premises--in the proof.

Here are some examples:

1. The so-called theorem of addition should be formulated thus:

"If the sine and cosine of any two angles α and β are known, the sine and cosine of the sum $(\alpha + \beta)$ of these angles can be found by the formula:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\ ;$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\ ."$$

Thus the values α, β and $(\alpha + \beta)$ of the angles themselves are not given, and there is no need to know them; these values are not given any attention in the proof of the theorem. However the theorem may be proved, all cases that may arise should be examined, including the border cases:

$$\alpha = 0,\ \beta = 0;\ \alpha = 0,\ \beta = \frac{\pi}{2};\ \alpha = \frac{\pi}{2},\ \beta = \frac{\pi}{2};\ \text{etc.}$$

2. In the conclusion of the formula

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad (1)$$

it is necessary to stipulate that it is established under the conditions

$$\alpha \neq (2k + 1)\frac{\pi}{2},\quad \beta \neq (2k + 1)\frac{\pi}{2},\quad \alpha + \beta \neq (2k + 1)\frac{\pi}{2}, \qquad (2)$$

and in proving it we must show precisely where we use these conditions. To emphasize that these conditions are organically connected with the theorem, it must be formulated thus: "If condition (2) is satisfied, then formula (1) is true." This is applicable to any other formula whose left or right part could lose its numerical sense.

5. Finally, let us dwell again on one method of instruction related to the inductive method (once again, it is insufficiently substantiated). Here is an example.

Wishing to teach the pupils the method of factoring polynomials by a proper grouping of their elements, the teacher would have difficulty beginning with the formulation of a general rule for using this method; only by solving a sufficient number of various problems illustrating all typical cases that may appear will the pupil be able to master fully the method from the theoretical and practical point of view and thus almost completely exhaust the content of the topic.

This method cannot be called inductive in the strict sense of the word. It is distinguished from the inductive method by the fact that it contains no generalizing stages, that is, it does not lead to the establishment of some general proposition. However, this method of communicating specific knowledge and skills is especially valuable in realizing the concrete inductive system of instruction. The teacher uses this method whenever he considers it expedient to preface the study of theory with the assignment of a group of exercises related to it.

THE RELATION BETWEEN MATHEMATICS INSTRUCTION AND LIFE*

G. G. Maslova and A. D. Semushin

The close relationship of education to life and work should be an important element in the content, organization, and methods of teaching mathematics. The abstractness of mathematical propositions does not in itself mean that mathematics is divorced from real life and practical experience. On the contrary, mathematics is valuable as a subject of study because--as a science of the spatial forms and quantitative relationships in the real world--it reflects and describes a large number of physical principles in an extraordinarily general form and facilitates their application to extremely diverse practical questions.

The connection with the practical world is broken when the instruction fails to reveal adequately the various ways in which the material studied can be applied, or when the pupils are unable to use the mathematical concepts they have acquired. When teaching subjects rich in practical application, one must emphasize content, and not just formal structure, or else mathematics instruction becomes divorced from life.

The mathematics curriculum for each grade includes work on measuring and graphic computations to be performed on data obtained by measurement (making simple estimates, constructing diagrams and graphs, solving practical problems, etc.). It also includes making models, working with calculating instruments and tables, solving problems illustrating the applications of mathematics to a variety of fields, and doing practical work in a locality.

Bringing school mathematics closer to life, however, cannot be limited to fulfilling certain specifications of the curriculum. Throughout any course the teacher must show the unique way in which the laws of mathematics reflect the real world; he must develop the pupils' ability to express practical work in mathematical form; he must develop in them abilities and skills that are needed in socially useful and productive

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*From Semushin, A. D. (Ed.). *On Teaching Mathematics in the Eight-Year School*, Moscow: Academy of Pedagogical Sciences of the RSFSR, 1961, pp. 9-26. Translated by Nancy Goldberg.

work, and he must work systematically at bringing together the school's methods of solving problems and the methods used in industry.

Let us go on now to a more detailed examination of these problems.

Showing How Mathematics Reflects the Real World

One important means of bridging the gap between mathematics instruction and life is to develop in the pupils a proper conception of mathematics as a science concerned with quantitative relationships and spatial forms in the world of objects. The children must be shown, with well-chosen examples, how the mathematical principles that they are learning reflect reality.

For example, as concepts are formed in the first stages of schooling, each concept must be the result of abstraction from concrete properties of objects and phenomena known to the pupil from his everyday experience. Here one should use the pupil's experience in school shops and his participation in socially useful activities; material that illustrates life in his village, district, city, or region; and information about the achievements of modern science.

Concrete images of mathematical concepts must be used not only during initial familiarization with these concepts, but throughout the entire period in which they are studied. This makes it possible for the pupils to recognize that, in studying a mathematical principle they are also studying many characteristically diverse phenomena of reality.

Thus, in studying the proportional relationship $y = kx$, the pupils should learn that the function $y = kx$ expresses a general type of relation that may exist between several quantities--for example, between the distance s travelled by a body at constant velocity v and the time t of its motion ($s = vt$); between the cost s of a quantity of identical items and their number; between the circumference C of a circle and its diameter D ($C = \pi D$, or $C \approx 3.14D$); between the linear (circular) velocity v in meters per minute of a part D mm in diameter put on a lathe, and the number n of revolutions of the spindle in one minute ($v = \frac{\pi Dn}{1000}$); between the mass of a body P and its volume V ($P = Vd$, where d = density); between the power P and the magnitude of the current I under constant voltage E ($P = IE$).

In using these examples one should stress that although in the function $y = kx$ the argument x (and therefore the dependent variable y) can assume any real value, for concrete physical examples the values of the argument and the values of the function are often limited. Thus, the mass of a body and its volume, the diameter of a shaft and the circumference of its cross section are expressed in positive numbers; the size of the shaft's diameter is limited by the dimensions of the lathe; and so on. Consequently, a graph of the relationship between two proportional quantities is often only a part of the graph of $y = kx$. The best way to illustrate this is to examine the graph of the function $y = kx$ and then to construct a graph of the ratios of several proportional quantities for cases where the variables cannot assume all real values.

Similar work can be done when studying the inversely proportional ratio $y = \frac{k}{x}$. It should be noted, however, that for examples of various ratios--inversely proportional ratios in particular--the selection of dependent and independent variables must be in keeping with the physical nature of phenomena. Thus one should not say that from Ohm's law, $E = IR$, it follows that the resistance R of a wire at constant voltage E is inversely proportional to the current I, which would seem to come from the equation $R = \frac{E}{I}$. In fact, since E and I are not independent, the size of the resistance is directly dependent only on the dimensions and the material of the wire.

In studying the linear relationship $y = kx + b$, the pupils should be shown that this function expresses the connection between various concrete quantities, in particular, the relationship of the final velocity v_t of a body and the time (t) of its motion at the constant acceleration described by $v_t = v_0 + at$, where v_0 is the velocity at time t_0, and a is the acceleration.

The number of examples of linear relationships that one may use in the eight-year schools is rather limited. In the upper grades, therefore, supplementary examples must be examined for the pupils to see clearly that the formula $y = kx + b$ expresses, in general form, the relation between a large number of quantities--for example, between the length ℓ_t of a rod and its temperature t, $\ell_t = \ell_0(1 + \alpha t)$, where ℓ_0 is the length of the rod for a fixed temperature taken to be 0°; and between the pressure P_t and the temperature t of a gas at a constant volume, $P_t = P_0(1 + \beta t)$, where P_0 is the pressure of the gas at a fixed temperature taken to be 0°.

As a result of this work, the pupils should understand that when studying various types of ratios between two quantities in mathematics, we are interested in the type and properties of the ratios (Is the ratio linear, quadratic, or of some other type? How does the given function change? What is its domain or the range of its values? etc.), but not in the quantities themselves.

This same idea of the uniqueness of the reflection of mathematical laws in the real world should also be brought systematically into the geometry course. The geometry curriculum in the eight-year school provides that the pupil become familiar with the most important relationships of the elements of a number of geometric figures (solid and plane) and with formulas for calculating surfaces and volumes of the basic solids. In using this material the teacher should show the applicability of the derived formulas to the solution of diverse practical problems. The pupils must understand, for example, that the formula $s = r\sqrt{2}$ can be used to solve such problems as finding the sides s of the largest cross section of a square beam that can be formed from a cylindrical bar of given radius r; determining the diameter of a metal core for making a bolt with a square head of given dimensions; and others.

For this same goal to be achieved in the arithmetic course and in the study of the elements of algebra, the pupils can compose formulas for solving problems and can compose problems based on a given formula. Through such work they will see that the formulas for solving problems in various subjects are often identical, but that, on the other hand, from one formula (such as that for the arithmetic average $\frac{a + b}{2}$) one can compose problems perhaps even externally unlike each other but whose mathematical content is the same. In physics any formula expresses an interrelationship with concrete quantities, while this need not be so in mathematics.

New material should be studied in close conjunction with the practical; new concepts, rules, and theorems should be introduced, where possible, with practical applications. (Here "practical" is used rather broadly, to include use in the various branches of mathematics as well as in the other sciences.) Today, practical problems are often treated in studying the elements of geometry and trigonometry, but it is rare for this sort of groundwork to be laid in an algebra course. By treating the practical more often, it should be possible to extend the applicability of such an approach to other sections of the secondary school mathematics curriculum.

One way to reveal the link between mathematics and everyday life is to give the pupils practical problems at the beginning of a new topic or section. The use of accessible examples helps to disclose the practical value of the topic or section and to define its place in the general system of the course. Sometimes simple problems with practical content can be given, and one can explain the basic mathematical content of a subject by using their solutions as examples. The pupils are thus shown the usefulness of studying the appropriate material. In the seventh grade, before the circumference of a circle is studied, questions such as these may be asked:

> How can we find the distance covered by a wheel during one revolution?
>
> How can we determine the distance to the water in a well from which a bucket is raised on a chain by means of a windlass?
>
> How far would such a bucket drop in three turns of the windlass?

Before the volume of a pyramid is studied in the eighth grade, problems can be given on finding the volumes and weights of bodies, the capacity of ships and of structures having the shape of regular pyramids.

Occasionally it is good to introduce the pupils to new material by using their own experience. As sixth graders examine various devices for constructing perpendicular and parallel lines, one needs to consider the skills they may have acquired when making these constructions in the shops. When axial symmetry is taken up in the sixth grade, it is necessary first to ascertain what kinds of figures having an axis of symmetry the pupils have made in the shops, whether the symmetry of a part was essential in constructing it, and how its manufacture was carried out. Finally, for this work to be successful, the teacher himself must find out beforehand how marking is done in shops or in factories and how various constructions are carried out in actual drafting and he must consider which of these methods can be shown in class and used in practical work.

In the study of triangles the pupils should be made aware--by their manufacture of various objects in the shops--of the use of the "rigidity" of the triangle in construction, in technology, and in everyday life. They should understand that the "rigidity" of construction is ensured by introducing slants in them, thereby forming triangles. An elementary example of this property's application is an ordinary gate (Figure 1). The pupils should be shown that tests for the congruence of triangles define their "rigidity."

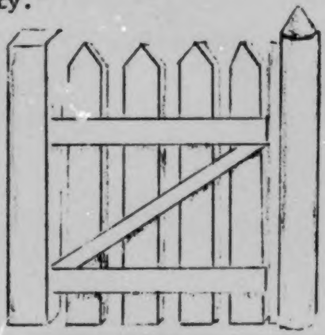

Figure 1

In the seventh grade the pupils become familiar with examples making practical use of the moveable quality of figures. Many examples can be considered when the parallelogram is studied. Here too the pupils' shop experience should be used.

In studying the similarity of figures in the eighth grade, the pupils learn that the properties of similar figures are applied extensively in drawing maps and plans to various scales in finding distances in marking parts according to diagrams in industry and so forth. The pupils must thoroughly understand the connection between scale and the coefficient of similitude.

Developing the Abilities and Skills Essential in Everyday Life and in Socially Useful, Productive Work

Mathematics instruction presents good opportunities for developing the pupils' abilities and often the skills essential for socially useful and productive work. The development of these skills should be carried out systematically throughout the pupils' schooling. We should give considerable attention to developing computational skill (oral and written), including work with elementary calculating instruments and skill in operating with approximate numbers (rules for rounding off, rules for reckoning digits). A significant number of exercises using this material should involve problems with practical content--especially problems in which the data are a result of measurements made by the pupils themselves. In solving these problems, first under the teacher's guidance and then on their own, the pupils determine what the accuracy of the result should be and acquire the ability to work with the most important measuring instruments--the scale rule, the protractor, the vernier calipers, and the transversal scale--while simultaneously becoming familiar with the mathematical principles underlying them.

Teaching calculation on the abacus is begun in the fifth grade. To consolidate and develop skill with the abacus, we should teach the pupils to use such calculation in later years as well (for example, in making estimates on the cost of repair of a building and solving other problems with practical content).

The systematic use of tables (of reciprocals, square roots, trigonometric functions, the circumferences of circles in terms of their diameters, and others) in various calculations will also help to develop skills that will be valuable in preparing the pupils for practical activity.

In the eighth grade, instruction in the slide rule is accompanied by its use in the solution of many problems in the algebra course and then by various calculations in mathematics, physics, chemistry, and other subjects. The pupils should be aware that the accuracy attained with the slide rule is sufficient for solving many practical problems. When the slide rule is studied, the pupils are taught to estimate a numerical result and to make visual estimates of the lengths of segments, the sizes of angles, and the areas of geometric figures.

The development of computational skill is supplemented by instruction in various techniques for making calculations and solving problems. For example, in calculating the area of a quadrilateral plot of land drawn to a certain scale, we first find the area of the quadrilateral ABCD represented on the map and then the area of the plot of land itself.

The area of the quadrilateral is equal to the sum of the areas of two triangles, ABC and CAD. It is simplest to choose a diagonal, such as AC (whose length we call a), as the base of both triangles and to draw altitudes of triangles from the vertices B and D, of lengths h_1 and h_2, respectively. Then the area would be

$$S = \frac{1}{2}ah_1 + \frac{1}{2}ah_2.$$

It is more convenient to do the calculations after this expression has been simplified.

$$S = \frac{1}{2}ah_1 + \frac{1}{2}ah_2 = \frac{1}{2}a(h_1 + h_2).$$

This example can show how a clever choice of the bases of the triangles and the use of algebraic manipulation simplify the problem's solution.

In studying functions, which have a wide practical use, considerable attention must be given to developing skills useful to the pupils in later work. Thus they should be able to "read" graphs of functions (that is, to find the value of a function for a given value of the argument, or the value of an argument for a given value of the function). For example, they should be able to estimate the square and cube roots of a number by using graphs of the functions $y = x^2$ and $y = x^3$, and they should be able to find the distance s travelled by a body moving at a velocity v for a time t from the graph of $s = vt$.

Complete mastery of the idea of functional relationship involves much preparatory work through the construction of various types of graphs in the arithmetic course. To this aim all types of numerical data may be used--data on the fulfillment of annual and quarterly production plans, on rural economy, and on the achievements of one's territory, region, city, or district. Diagrams can be made to reflect school life (the amount of scrap metal and waste paper collected by Pioneer teams, the number of trees and bushes planted by the children, the use of electric power in the school at various times of day, and so on). The pupils' experience in other classes should also be utilized. For example, laboratory work in chemistry, physics and biology can provide data useful for practice in reading and constructing graphs; and the school shops are often a valuable source of factual material on direct and inverse proportions.

Problems on finding parameters of equations from graphs of these equations are useful in developing mastery both of the abstract concept of a function and of concrete practical skills. For example, the pupils might be asked to determine the values of a, b, c, d, and k from the graphs in Figure 2. In addition to the prepared graphs in mathematics lessons, graphs made by the pupils themselves and based on the results of laboratory work in science classes can be used.

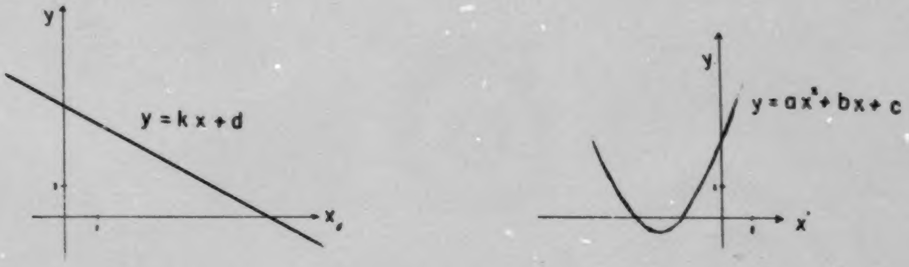

Figure 2

In their mathematics course the pupils should become familiar with calculations from given formulas. This makes it possible to increase the number of practical exercises and helps to develop the pupils' computational skill. Exercises of this type can involve finding the numerical value of an algebraic expression and, in the geometry course, calculating volumes and surfaces of solids from formulas. For example, in studying the volumes of bodies symmetric about some axis of rotation, the pupils can be given a problem on calculating the volume of a barrel from the formula:

$$V = 0.785h\left(\frac{D + d}{2}\right)^2,$$

where V = the volume of the barrel, D = the diameter of its widest part, d = the diameter of its narrowest part, and h = the height of the barrel.

In geometry in the eight-year school, the pupils should be shown how to use the construction devices that have the broadest practical application (in drafting and marking), including the straightedge, the compass, the drafting triangle, and the bevel.

The techniques involved in especially important constructions (constructing parallel and perpendicular lines, bisecting a segment or an angle) should be so firmly grasped that when the pupils later solve construction problems, they will be able to concentrate on seeking the problem's solution, rather than on the external form of the problem. To develop skill in construction, the pupils must solve construction problems effectively, that is, the solution they obtain must be such that it can be carried out to a sufficient degree of precision using actual instruments. Since many specialized instruments used in practical work operate on principles that the pupils can grasp, they should be familiarized with these instruments whenever possible—especially if these instruments will later be encountered in the shops.

It should be interesting to the students to substantiate geometrically not only the correctness in form of parts made in the shops, but also the accuracy of the various devices used to check this correctness. Thus, in the carpentry shop the parallelism of the opposite sides of a frame may be verified by the perpendicularity of its adjacent sides. In this example the pupils can confirm the accuracy of the checking device by themselves.

The pupils should also be acquainted with elementary devices for the approximate solution of construction problems like dividing a segment, or an arc of a circle, or an angle, into equal parts, and rectifying a circular arc. These problems are simple enough in themselves and are often encountered in practical work.

Towards a Closer Connection between the Methods Used in Problem Solving in School and Those Used in Practical Work

Children should be prepared to participate in socially useful and productive work by perceiving a close connection between the methods of solving problems in school and those used in industry, agriculture, and everyday life. The idea of functional relationship is used in almost any area of modern industry--diagrams, graphs, nomograms, and graphic methods of problem solving in general, both approximate and exact. Nevertheless, these methods of problem solving have not been sufficiently discussed in secondary-school instruction (grades 5 to 8).

Instruction in geometry can be brought closer to the requirements of industry by reorganizing the traditional system for teaching the methods of solving construction problems. It has been stated more than once that the "classical" manner of solving problems with compass and straightedge does not take into consideration the requirements of everyday activity and does not prepare pupils for practical work. In actual practice, construction problems are solved using not only the compass and the straightedge but also the set square, the protractor, and various equivalents of the bevel (an instrument that permits an angle to be constructed equal to a given one, without any auxiliary constructions). The mathematical aspect of solving construction problems with a compass, straightedge, set square, and bevel can and should be formulated just as rigorously as in constructions using only a compass and straightedge. In the first geometry lessons in the sixth grade, before this aspect has been formulated, one can permit the use of the ruler as a scale rule and of the protractor as an instrument for measuring angles. Later the ruler should be used only as a straightedge, while the protractor is used for constructing angles equal to a given one (that is, it is used as a bevel).

With the use of these techniques, constructions with compass and straightedge in the school become a realistic example of solving construction problems using limited means. Construction problems should still be solved in the "classical" manner, that is, the validity of all constructions should be rigorously proven. The complete solution of these problems, therefore, involves four stages: analysis, construction, proof, and investigation.

The pupils should thoroughly understand that, depending on the conditions and purposes of the constructions, a single construction can often be made with different instruments and, consequently, by different methods. Thus, perpendicular lines are often constructed with the drafting triangles and the straightedge, or with the T-square and the drafting triangle. In marking plane surfaces that have protuberances so that the triangle cannot be employed, the compass and straightedge are used for constructing perpendicular lines.

Using many instruments for constructions and teaching the most practical methods of construction makes it possible to acquaint the pupils with graphic methods of solving geometry problems that are usually related to problems on calculation. The use of graphic methods significantly increases the number of problems whose solutions are within the pupils' reach, reinforces their ability to make constructions, develops their accuracy, increases their attention span, and provides them with means of checking an analytic solution. The graphic method is often simpler than the analytic one and may still provide the necessary accuracy. It is therefore often used in performing the most diverse calculations.

Let us give some examples.

Suppose that one must solve a triangle given certain elements—that is, suppose that one is given some of the sides and angles of a triangle and that one must find the others. This problem is usually solved analytically, but the pupils should also be shown a graphic solution, especially since it is very simple if the construction of the triangle from the given elements has caused no difficulties. For a graphic solution of this problem a triangle must be constructed from the given elements, then the unknown elements measured (the triangle is usually constructed to a specific scale).

Let us look at another example. Say we have to measure the distance between two points A and B, separated by an obstacle, and suppose that the measured quantities are the distances AC, BC, and the angle ABC (point C is chosen arbitrarily). The most logical device for solving the problem is the concept of the similarity of triangles: From the given measurements, a triangle $A_1B_1C_1$ is constructed similar to triangle ABC, and side A_1B_1 is measured off. The unknown distance is obtained if we multiply the length of the segment A_1B_1 by the coefficient of similitude. In determining the length of the slants of a structural frame (truss) (Figure 3) it is also useful, instead of using right triangles and the Pythagorean theorem, to determine the unknown length graphically, using the similarity of triangles. The graphic method should also be

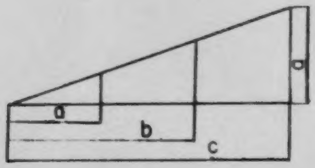

Figure 3

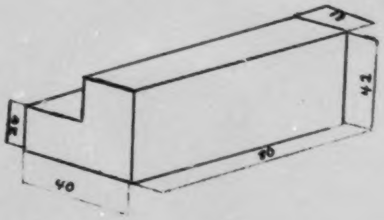

Figure 4

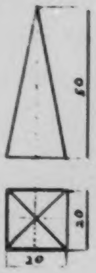

Figure 5

used more often in solving equations and systems of equations. This helps greatly in consolidating the abilities and skills used in constructing and reading graphs.

In the eight-year school much attention must be given to developing the pupils' spatial imagination. This should be done not only by thorough study of the relationships between the elements of plane and solid figures, but also by a systematic use of stereometric images and concrete bodies in the study of plane geometry. Thus, in studying the areas of plane figures, the pupils can be asked to calculate, from the dimensions given on the drawing, the lateral surfaces and the entire surface of a solid composed of right parallelepipeds (Figure 4). Similar problems should be solved using models of solids and other objects, with the pupils doing all the necessary measuring by themselves.

Occasionally it is useful to have geometry problems solved with prepared drawings given in orthogonal projection (find the entire surface and lateral length of the edge of the pyramid in Figure 5) and familiar to the pupils from the drawing course. This is all the more important since orthogonal projection (composite drawing) is a fundamental method of representation very widely used in technology. The ability to recreate a body from its projections is thus a very important type of spatial imagination for pupils to develop.

Developing the Ability to Give Mathematical Form to Practical Problems

A basic means for developing such ability is to teach problem solving according to the current curricular material. Before selecting problems to be solved with the pupils, however, one must establish new criteria. In mathematics lessons a certain amount of attention must be paid to solving problems of an applied nature--problems with industrial content from various subjects in school, problems one must cope with in everyday living. Moreover, the content and number of these problems should not distract the pupils from studying mathematics proper.

The need for an examination of applied problems in the mathematics course has already been determined in work experience. The mathematical content of these problems should be clearly expressed, the applied aspects of the problem should be accessible to the pupils, and the situation described in the problem should be contemporary and realistic. The simplest example of such a problem is the estimation of expenditures for various kinds of repair work (apartment decorating, electrical wiring, simple construction work, etc.).

A significant number of problems with practical content have been published in periodicals and in various collections. When the teacher uses these problems, he must acquaint the pupils with both exact and approximate solutions. It is useful to show that the approximate nature of the answer can be produced not only by using the problem's initial data in approximate form, but also by the limited accuracy required in the answer as determined by the specific conditions of the problem.

For example, in the first two problems given below, the data are approximate. In the first, the diameter of the cable (or rope) on which the load is suspended is not taken into consideration; in the second, the piece is assumed to have the shape of a right parallelepiped, while in reality the sides have large radii of curvature and there are large tolerances for the linear dimensions.

> Problem 1. The drum of a winch was 400 mm in diameter and made five rotations. How far was the load raised when this happened?
>
> Problem 2. From a piece of steel whose right cross section was 200 mm by 400 mm and whose length was 2m, a rolled metal plate was obtained whose cross section was a rectangle with sides of 100 mm and 250 mm. What was the total length of the rolled metal if the metal loss is assumed to be 2%?
>
> Problem 3. Determine the length of a cylindrical spring with an internal diameter of D mm if the diameter of the wire is d mm and the number of spirals of the spring is n.

Using the last problem as an example, the pupils can be shown how a number of assumptions are introduced in solving a practical problem. Here we make the approximation that the length of each spiral of the spring is equal to the circumference of a circle of diameter $D_{av} = D + d$; and, a result of this assumption, the unknown length ℓ of the piece can be found rather easily from the formula $\ell = \pi D_{av} \cdot n$.

At the beginning of the school year the mathematics teacher should become acquainted with the trade and polytechnical school curricula and should learn what mathematics knowledge the pupils will need, and where and to what extent the pupils will use this knowledge. It is useful for the teacher to select in advance the types of problems that the pupils may encounter in trade lessons, when studying related disciplines, and in everyday life. Some of the most interesting problems can be examined with the pupils during their mathematics lessons. Practical work in the classroom and measurements in the field can help to develop the pupils' abilities to give mathematical form to the practical problems of everyday life.

The content of practical (applied) work is determined by the curriculum of the eight-year school. Practical work should be conducted throughout the school year in conjunction with the study of the appropriate curriculum material. The time devoted to this work can vary from 10- or 15-minute discussions (for example, at the end of a lesson) to special, separate lessons--depending on the complexity of the assignment.

In applied work, great use can be made of objects in the immediate surroundings (measuring the length and width of a room, calculating its area and volume, finding the perimeter and area of a pupil's notebook, of a tabletop, etc.). Each school should also have appropriate materials available for distribution. For example, the fifth grade might have sets of rectangles and triangles; the seventh grade--sets of right parallelepipeds, right prisms, cylinders; the eighth grade--sets of pyramids. cones, spheres, and combinations of polyhedra and round bodies.

These sets can be made by the pupils in industrial art lessons or as homework assignments. The sets should include both elementary manufacturing parts and material distributed in the drafting course. In order to check the accuracy of the pupils' practical work, models of sets can be provided to the teacher, and charts can be drawn up for each of them that indicate the dimensions of the items to be made. Answers to all written problems should also be provided.

Materials from industrial field trips may help the pupils formulate and solve problems whose content reflects practical requirements; and manual training lessons can help them acquire some skills in applying mathematics to the solution of practical problems. In such lessons the pupils can apply their ability to solve construction problems (for example, in marking pieces of work) and their familiarization with drawings and representations and with functions and their graphs. Special attention, however, should be directed to drilling their skills in approximate calculation.

When the same type of problem occurs in several subjects, the teachers must be careful to use uniform terminology and methods in dealing with these problems. For example, problems on percentage and on proportions in chemistry and physics classes must be solved in the same way as those in mathematics classes.

Eliminating the gap between mathematics and everyday life is important in preparing the pupils for practical work. At the same time, revealing the connections of the eight-year-school mathematics course with life, with work in the shops, with socially useful work, and with related school subjects is one way of demonstrating the abstract nature of mathematics and of improving the pupils' mathematical sophistication.

THE PUPIL'S ACTIVITY AS A NECESSARY CONDITION FOR IMPROVING THE QUALITY OF INSTRUCTION*

I. A. Gibsh

The school reorganization that is currently under way assumes that new principles of teaching the basic sciences will help to meet the new demands of educators. Of these principles, the most important is that the whole educational process should be based on a method that would act in every way possible to awaken the pupil's interest and activity.

It is obvious that the soundest knowledge is that acquired through active thinking -- through having independently found a method of solving a problem and having generalized it as far as one can -- and that the facts thus established can always be reproduced by the person who has once found them. Moreover, the methods of reasoning and the conclusions drawn by a pupil in solving a problem can be applied to the solution of other questions, thus broadening his experience and helping him to seek means of solving problems of ever-increasing generality.

Such a degree and such a level of pupil development will assure the pupil of a substantial education in the basic sciences in general and in mathematics in particular. The knowledge he acquires on his own and his ability to pose, solve, and investigate a question will be his most valuable achievement not only in mastering fundamentals of the sciences, but in all his productive activity in school and after he has finished school.

Observations of the pupil's approach to this activity confirm that such activity is most productive when it is done in close conjunction with the solving of problems that interest the pupil--assignments that permit him to show his independence and to display his personal creativity and talent when he solves them. This is why instruction that increases the pupil's interest in the subject in every possible way and utilizes his creative abilities, is the most crucial of the tasks facing the new school and is justifiably considered the school's primary goal. The teacher who recognizes the extreme importance of making the educational process active will strive to effect this activation in every possible way. He has at his disposal many methods with which to do this, the most effective of which are the following.

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*From Semushin, A. D., (Ed.). On Teaching Mathematics in the Eight-Year School, Moscow, Academy of Pedagogical Sciences of the RSFSR, 1961, pp. 26-56. Translated by Nancy Goldberg.

The Statement of the Question Forming the Content of a Chapter or Topic

The exposition of each section and, wherever possible, of each topic of the school mathematics course should begin with a statement of the fundamental question to be answered therein. This serves as a short introduction to the chapter, establishes the connection with the preceding material, and explains the basic purpose of the topic—the problem that is to be solved. This introduction opens vistas to the audience, awakens their interest in solving the problem, and often outlines general methods for the solution itself.

The question can be stated in two forms, which are not sharply differentiated.

1. The teacher may quickly bring the pupils into contact with the concept or small sphere of new concepts, without prefacing his discussion with more general observations.

Example 1. The unit on "Triangles" is best begun by explaining that the triangle differs from all other polygons in its special property of "rigidity," which can be demonstrated by comparing a closed polygon of more than three sides with a closed triangle. The first is subject to deformation that changes the angles while retaining the sides, whereas the second is not subject to this deformation. The geometric expression of this fact is then found, and it turns out that the equality of the corresponding angles of two triangles is a consequence of the equality of their corresponding sides. Here one can establish (and later develop) the question of the possibility or impossibility of a converse conclusion whose answer leads to the idea of similarity.

Example 2. It is advisable to preface the units on "The Segment Joining the Midpoints of Two Sides of a Triangle" and "The Segment Joining the Midpoints of the Two Nonparallel Sides of a Trapezoid" with a question such as "Let two lines AE and A_1E_1 be given in a plane. Equal segments AB, BC, CD, DE are marked off on line AE, and through their end points mutually parallel lines are projected until they intersect line

A_1E_1 in points A_1, B_1, C_1, D_1, E_1. Will these segments A_1B_1, B_1C_1, C_1D_1, D_1E_1 be mutually equal?"

Each pupil is asked to make a drawing in his notebook and to answer the question experimentally by measuring segments A_1B_1, B_1C_1, C_1D_1, D_1E_1. It is thus shown that, even though AE and A_1E_1 might be at various angles to each other and despite the various lengths of the equal segments on line AE (due to the various lengths of the drawings of line AE itself), the corresponding segments made by the parallel lines on A_1E_1 will be mutually equal. The need arises of proving this assertion. The teacher proposes that a particular case be examined as a preliminary: when the given lines intersect at some point 0 and when congruent segments are marked off beginning from this point 0.

The question of the converse theorem's validity may be asked. It is explained that the converse is true only if the equal segments are marked off from the point of intersection 0 of the given lines. From here, there is a direct transition to the theorems on segments joining medians.

Example 3. To decide on the number of points that determine a circle, the pupils must first find answers to these questions: (a) How many circles can be drawn through a given point? (b) How many circles can be drawn through two given points? How can this be done? Where are their centers located?

After answering these questions, the pupils will easily determine how many circles can be drawn through three given points. It remains for the teacher to explain the necessary and sufficient conditions for producing a circle through three given points, to establish the reason for the necessity and sufficiency of these conditions, and to formulate the result in a sentence: "A circle is completely determined by three points that do not lie on the same straight line," emphasizing that the term "completely determined" has already been encountered in application to the straight line and that the sentence formulated with this term simply means that "one and only one circle may be drawn through three points that are not on the same straight line." The teacher may if he wishes make the indicated formulation in reverse order, but it is important that the pupils express this notation precisely in order that they learn the material thoroughly.

Example 4. The theorem on a diameter perpendicular to a chord arises as a natural consequence of the fact that triangle OCD, which joins the center O of the circle with the endpoints C and D of the chord CD, is isosceles, so that the theorem on the coincidence of the altitude of an isosceles triangle with its median and bisectors is applicable.

2. The second way of stating the question consists in revealing its essence before hand, in order to illuminate the question from various general positions at the very beginning. This introduces the pupils to the entire sphere of concepts and ideas that constitute the new material and serves as a preparation for the most natural and satisfactory perception of this material.

Example 5. The unit on "The Circle," studied in the seventh grade, is prefaced with the following general conceptions. At first the pupils, using their intuitive concept of the circle as a basis, can compare a circle with a straight line and ascertain the similar and the distinctive features of these curves: (a) both the circle and the straight line divide a plane into two regions, each containing points located on different sides of the boundary; a segment joining a point in one region with a point in the other region must intersect this boundary; that region to which the center of the circle belongs is usually called its interior, and the other region is called its exterior; (b) a circle is a closed curve. If, in going from some point A in the circle, we move along it constantly going in the same direction, we will return to the original point A; this is what we mean when we say that a circle is closed. It is obvious that a straight line does not have this property, it is not a closed curve.

These two properties of a circle, however, do not determine it. Any convex polygon divides a plane into two regions—interior and exterior—and it too has the property of being closed. The circle has a property that no other closed curve has, however. It is defined as a closed curve (in a plane) all of whose points are located at precisely the same distance from a given point called the center. This definition can be made more precise by showing that on each ray radiating from a given point O there is one and only one point whose distance from O is equal to a given segment OA. Rotating the segment OA around its origin O, we see that its endpoint A coincides in turn with each point which is at

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the given distance 0A from point 0; the union of all these points is a circle with center 0 and radius 0A.

Example 6. The following discussion may serve as an introduction to the unit on "Features of Similar Triangles." Having drawn an arbitrary convex pentagon ABCDE, the teacher draws from some point A' in the plane a segment A'B', parallel to side AB and equal to, for example $\frac{1}{2}$ AB; then from point B' he draws a segment B'C', parallel to side BC and equal to, say, $\frac{2}{3}$ BC; and finally from point C' he draws a segment C'D', parallel to side CD and equal to, say, $\frac{3}{4}$ CD. If he next draws from points D' and A' half-lines D'E' and A'E' — which are parallel to sides DE and AE, respectively -- to their mutual intersection at point E', then pentagon A'B'C'D'E' is obtained whose angles are correspondingly equal to the angles of pentagon ABCDE, but whose sides are not proportional to the sides of the latter since

$$\frac{A'B'}{AB} = \frac{1}{2}; \quad \frac{B'C'}{BC} = \frac{2}{3}; \quad \frac{C'D'}{CD} = \frac{3}{4}$$

The pupils' attention is directed to the fact that in the case of a pentagon only the lengths of three of the sides of polygon A'B'C'D'E' can be chosen at will, since the lengths of the last two sides are already determined by their direction.

The same thing can be seen in constructing a quadrilateral A'B'C'D' whose angles are correspondingly equal to those of a given quadrilateral ABCD but whose sides are not proportional to the sides of the latter. Again the lengths of the last two sides are determined by the choice of the other sides.

Another conclusion must be drawn in the similar construction of a triangle A'B'C' whose angles are correspondingly equal to the angles of triangle ABC. Here, having chosen the length of side A'B', we are forced to stop since the lengths of sides B'C' and A'C' are already determined by their direction, and point C' will be the point of intersection of half-lines B'C' and A'C'.

The question arises whether the lengths of the sides of triangles A'B'C' and ABC would not be found to be in some ratio with one another. A theorem is stated and proved. "If the angles of one triangle are correspondingly equal to the angles of another triangle, then the sides of these triangles are proportional." Then the pupils compose and prove formulations of the converse theorems expressing the second and third features of similar triangles.

Example 7. When the pupils are introduced to the essence of the unit on "Trigonometric Functions of the Acute Angle," it is advisable to preface it with the following discussion.

a. It follows from the conditions for the congruence of two triangles established in geometry, that any three independent basic elements (sides or angles) of a triangle determine the remaining basic elements — which are thus functions of the three given elements. If the three independent elements of a triangle are given geometrically (that is, graphically), the remaining elements of the triangle can be found by geometric construction. But if the three independent elements of the triangle are given by numbers obtained by measuring these elements, the remaining elements must be calculated.

This calculation, however, can be made only if the unknown values are successfully related to the given data by a system of equations expressing a functional relationship. But no equation exists that expresses the relationships between angles and sides of a triangle by means of the ordinary functions the pupils know.

To solve this problem, special functions were introduced long ago that served as a unifying element between the angles and sides of a triangle and permitted the geometric method of finding the elements of a triangle to be replaced by the method of solving equations — the so-called analytic method. Just what functions are these?

b. If on one of the sides of an acute angle AOB (Figure 1) we take an arbitrary point C and construct its projection C' to the other side of the angle in a direction perpendicular to the second side, we simultaneously construct a projection OC' of segment OC onto side OB. If we

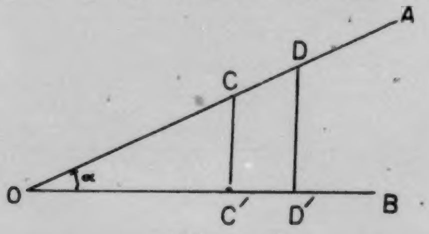

Figure 1

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make the same construction beginning at some other point D on side OA, we simultaneously construct a projection OD' of segment OD onto side OB. With a change of position of the original point on side OA, the projecting segments (CC', DD') as well as the projected segments (OC, OD) and their projections (OC', OD') will change. At the same time, however, each of these ratios will remained fixed:

$$\text{(a)}\quad \frac{\text{projecting segment}}{\text{projected segment}}, \qquad \text{(b)}\quad \frac{\text{projecting segment}}{\text{projection}},$$

that is,

$$\text{(a)}\quad \frac{CC'}{OC} = \frac{DD'}{OD}, \qquad \text{(b)}\quad \frac{CC'}{OC'} = \frac{DD'}{OD'} .$$

Thus each given acute angle uniquely determines each of the ratios (a) and (b). It is easy to prove the converse that each of the ratios (a) and (b) uniquely determines a certain acute angle. Therefore each of ratios (a) and (b) may be considered a new numerical characteristic of an angle; but this numerical characteristic of an angle is distinguished because it is not found by measuring the angle, but by measuring two segments connected with this angle and dividing one of the results of this measurement by the other. In view of the peculiarity of the connection of the value α of angle AOB with each of ratios (a) and (b), the examination and study of these ratios has proven especially fruitful, and they have been given special names. Ratio (a) is called the "sine of angle α," and ratio (b) is called the "tangent of angle α."[1]

c. After giving the pupils the above information and thus revealing the essence of the topic with sufficient thoroughness, the teacher asks the pupils: (a) to find experimentally (using measurement of the segments) the sine and tangent of an angle given geometrically; and (b) to find experimentally (by geometric construction) an angle, given its sine or tangent.

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[1] The proposed definitions of the trigonometric functions of an acute angle have the advantage that: (a) they stem from the examination of an angle and not of a triangle; thus the relationships between the elements of a right triangle are already the consequences of these definitions; and (b) they make possible a natural definition of the trigonometric functions of an obtuse angle.

The Heuristic Method of Instruction

The heuristic method of instruction consists in the teacher's asking the pupils, with an aim to establishing new concepts and facts, a sequence of carefully composed and ordered questions that the pupils must answer, thereby gradually developing for themselves the essence of the new concepts and facts. The effectiveness of the heuristic form of instruction depends primarily upon the extent to which the sequence of questions is composed correctly — both logically and methodologically — and on how adeptly the teacher conducts the lesson in heuristic form. Here the statement of the question is an organic part of the lesson (a lesson devoted to the communication of new material). Thus, the system of questions posed by the teacher should embrace this part as well.

Example 8. Unit on "Quadratic Equations."

Teacher: Who can write the simplest equation having 2 as a root?

Pupil: $x - 2 = 0$.

Teacher: Who can write the simplest equation having 2 and 5 as roots?

Pupil: $(x - 2)(x - 5) = 0$.

T: Correct. But if we multiply out the left-hand side of this equation, we may replace it by the equation $x^2 - 7x + 10 = 0$, which has the same roots. If you were given this latter equation, how could you prove that it has roots of 2 and 5?

P: You could factor the left-hand side of the equation and represent it in the form of $(x - 2)(x - 5) = 0$, and in this form it would be clear that it has 2 and 5 as roots.

T: And how would you factor the trinomial $x^2 - 7x + 10$?

P: We would view its middle term as a sum and rewrite the trinomial as the polynomial $x^2 - 2x - 5x + 10$, which we would factor by the method of grouping.

T: Correct: But can we factor any trinomial of this type in this way? For example, how would you factor these trinomials:

$$x^2 - x - 12, \quad 16x^2 - 16x + 3, \quad x^2 - 2x + 5?$$

P: $x^2 - x - 12 = x^2 - 4x + 3x - 12 = x(x-4) + 3(x-4) = (x + 3)(x - 4)$; and $16x^2 - 16x + 3 = 16x^2 - 4x - 12x + 3 = 4x(4x - 1) - 3(4x - 1) = (4x - 1)(4x - 3)$; but we don't know how to factor the trinomial $x^2 - 2x + 5$. Can it really be factored?

T: You have factored the first two trinomials correctly and should now be able to solve the equation $x^2 - x - 12 = 0$ and $16x^2 - 16x + 3 = 0$.

P: The first equation has roots -3 and 4, and the second has roots $\frac{1}{4}$ and $\frac{3}{4}$.

T: Correct. With regard to the third trinomial, it wasn't accidental that you couldn't factor it — it cannot be factored. Not all of you, however, were able to find the factors of the first two trinomials, and those who were able wasted time at it. The question arises of when a trinomial of the form $ax^2 + bx + c$ can be factored and whether some general method cannot be found to factor it when this can be done.

The teacher then acquaints the pupils with the method of forming a perfect square from a quadratic trinomial $ax^2 + bx + c$. In so doing, of course, he begins with particular cases and passes gradually to the most general case.

Example 9. From geometry it is known that a straight line is determined by two given points. Consequently, to each pair of points given in any form in the plane there corresponds one and only one straight line passing through these points. Let us consider that on a plane coordinatized by some system of axes points are chosen with coordinates (2,5) and (-1,-4). Let us try to determine the equation of the line passing through these points.

T: What form should this equation have?

P: We can look for this equation in the form of $y = ax + b$.

T: Correct. Consequently, to determine the unknown equation we should find its coefficients a and b. How can this be done?

P: We must use what we know about the equation's unknowns.

T: Correct. More precisely, we must subordinate the coefficients (make them subordinate) to the requirements that are made of them as the conditions of the problem. What are these requirements?

P: The unknown equation must be the equation of a straight line passing through points (2,5) and (-1, -4).

T: True. But how can we express this requirement algebraically?

P: We must write down that the pairs of numbers (2,5) and (-1, -4) are the solutions of an equation $y = ax + b$.

T: Go to the blackboard and write this.

P: $5 = 2a + b$, $-4 = -a + b$.

T: And what shall we do now?

P: Now we must solve this system of equations for the unknowns a and b.

T: Correct. Do it.

P: $3a = 9$; $a = 3$; $b = -1$.

T: Consequently, what equation satisfies the conditions of the problem?

P: $y = 3x - 1$.

This example acquaints the pupils for the first time with the method of undetermined coefficients. After the pupils have acquired some skill in applying this method to the solution of the same problem, with different points given, the teacher can ask them to solve the problem in a general form.

The use of these examples can help the teacher form a definite conception of the essence of the heuristic method of instruction. To prepare the best system of questions (which must be asked according to a given system), the teacher is advised to prove the statement or solve the problem in advance by means of a careful analysis, which is the most powerful means of attaining the indicated goal and which the pupils should master.

Assuming that the teacher is wholly familiar with the essence and means of applying the analytic-synthetic method [2;3], we shall limit ourselves to three examples.

Example 10. The altitudes of an acute triangle are extended until they meet a circumscribed circle. Prove that the segments of these lines between the point of intersection of the altitudes and the point of intersection with the circle are bisected by the appropriate sides.

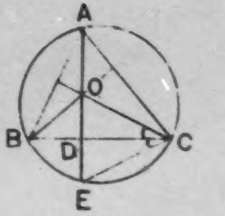

Figure 2

Analysis. If 0 is the point of intersection of the altitudes of triangle ABC (Figure 2), point D is the base of altitude AD, and the extension of altitude AD intersects the circle at point E, then the problem is to prove that OD = DE.

We make the following scheme of analysis:

1. OD $\overset{?}{=}$ DE.
2. ∠OCD $\overset{?}{=}$ ∠ECD ("If the altitude of a triangle is simultaneously a bisector, then the triangle is isosceles").
3. ∠OCD $\overset{?}{=}$ ∠BAD (both are complementary to angle B); ∠ECD $\overset{?}{=}$ ∠BAD (inscribed angles based on a common arc, BE).

The converse course of reasoning (3, 2, 1), that is, the synthesis, is the proof of the theorem.

Example 11. Prove that the product of two sides of a triangle is equal to the product of the altitude on the third side and the diameter D of the circumscribed circle.

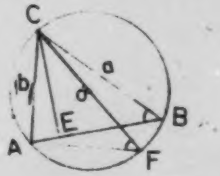

Figure 3

Analysis. Designating the foot of the altitude h_c by E (Figure 3), and by D, the diameter CF passing through vertex C of the triangle and the center 0 of the circle circumscribed about it, we make a scheme of analysis:

1) $a \cdot b \overset{?}{=} h_c . D$.

2) $a : h_c \overset{?}{=} D : b$.

3) Triangle BCE $\overset{?}{\sim}$ Triangle FCA.

4) $\angle B \overset{?}{=} \angle F$ (inscribed angles based on a common arc, AC).

Example 12. Prove that if the feet of the altitudes of an acute triangle are joined by straight lines, a new triangle is formed for which the altitudes of the first triangle are angle bisectors.

Analysis. If altitudes AD, BE, CF of triangle ABC intersect at point 0 (Figure 4), then it must be proved that DO, EO, FO are the bisectors of the corresponding angles D, E, F of triangle DEF. Let us prove, for example, that:

(a) $\angle FDO = \angle EDO$.

How can we prove this? Where should we begin? To answer this question, we must study the conditions and the drawing carefully. We note that angle FDO comprises side DO and diagonal DF of quadrilateral BDOF with two opposing right angles; in exactly the same way, angle EDO is formed from side DO and diagonal DE of quadrilateral CDOE with two opposing right angles. Consequently, to prove equality (a), it is sufficient to establish that one may circumscribe circles around quadrilaterals BFOD and CEOD, since then

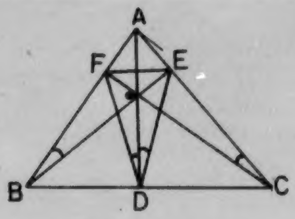

Figure 4

(b) $\angle FDO = \angle FBO, \qquad \angle EDO = \angle ECO$,

and to prove that

(c) $\angle FBO = \angle ECO$,

which follows from the fact that angles FBO and ECO are complementary to angle A.

The Teacher's Role in Developing the Pupil's Ability to Find Methods of Solving Problems Independently

It is obvious that the pupils' ability to find proofs of propositions and solutions to examples and problems independently must be encouraged by the teacher's use of a system of teaching and a methodology.

1. The teacher's primary job is to develop the pupil's ability to seek independently the answers to mathematical questions by the example of his teaching, giving prepared formulas and results and indicating methods of solving problems only in those relatively rare cases when it is impossible to expect the pupils to find the solution entirely on their own. Usually, however, the teacher can remark in advance on the natural, most logical course of solving a problem and can lead the pupils along this course during their active efforts in class to find proofs of propositions or to do exercises. This permits him to give many examples of how to seek solutions to problems, which should teach the pupils an appropriate methodical approach to independent work, instill good habits in them, and become a constant stimulus to their work, determining its character and direction.

2. A significant role in developing the ability to solve mathematical problems independently is played by the detailed explanation of generalizations and general instructions that can be made with regard to the application of the methods, devices, and means used to prove propositions and to work examples and problems. The teacher must not miss any opportunity to do this. It is just such general instructions that are able to help the pupil begin the necessary reasoning, to suggest to him the choice of the most natural, most expedient way--to hand him a thread.

In conjunction with this, the methodology of mathematics teaching has long recognized the necessity and advisability of ending certain sections and even topics of a course by generalizations of this type, by the use of some system of general instructions that are often put in the form of schemata, tables, summaries, and drawings. The following, for example, are usually presented in the form of a scheme: (a) the method of solving a first-degree equation with one unknown; (b) the method of investigating the first- and second-degree equations and systems of first-degree equations; (c) the method of evaluating trigonometric functions (by quadrants); and some other results.

There is far from enough of this, however. An essential element of the teaching of school mathematics should be for each teacher to compose (with the pupils' help, of course) general conclusions that would not only summarize the material systematically but that would also serve in the future as starting-points for the pupils' independent reasoning.

Example 1. As is known, in both algebra and trigonometry an important place is occupied by algebraic identities or transformations. Is it proper for methodological manuals on teaching algebra and trigonometry to contain no general instructions in this area, for them not to isolate the principles that should underlie the use of these identities, and for them not even to indicate general methods?

It is almost impossible to think of a book or even an article in which any space is devoted to the systematic use of algebraic identities. Only occasionally do authors group exercises according to the methods of solving them, thus isolating the methods and influencing the readers' acquaintance with the system of methods applied in solving exercises of each type. Thus in exercises concerning the proof of polynomial identities the following methods are indicated: (a) the method of transforming each side of an equality to an equivalent term; (b) the method of arranging both sides of an equality by powers of some variable x and subsequent comparison of the coefficients of the same power of x on both sides of the equality; (c) the method of shortening the given identity beforehand by replacing it with another that lends itself more quickly to being proved (for example, by placing similar terms on the same side of the equation).

Equipped with the knowledge, the pupil is no longer helpless in solving an exercise in this field; on the contrary, he will actively strive to use the most proper method known to him and will try to find or devise the best solution.

Example 2. The following general hints may be given to introduce some sense of order and to orient the pupils as they begin factoring polynomials.

1. First remove the common factor.

2. If a polynomial (after the common factor is removed) is (a) a binomial, (b) a trinomial, or (c) a polynomial of four terms, then investigate to see whether it may be (in case (a)) a difference of squares, or a sum or difference of cubes (in the upper grades—the difference of two elements raised to an even power, the sum or difference of two elements raised to an odd power); (in case (b)) the square of the sum or difference of two elements; or (in case (c)) the cube of the sum or difference of two elements (in these cases factoring can be done according to the appropriate formula).

3. If hint 2 does not apply, take any variable contained in the polynomial as the main one, arrange the polynomial by powers of this variable, and then factor it — grouping the elements in the order in which they were written.

4. If the device described in hint 3 does not help to factor the polynomial, then an attempt should be made to group the terms of the polynomial in another way, using as a basis: (a) their signs, (b) their coefficients, (c) their dimensions, (d) the possibility of applying multiplication formulas to some group of elements.

5. If a trinomial arranged in descending powers of some variable is not the square of a sum or difference, it is rearranged (in the seventh grade) by decomposing (splintering) its middle element into two summands or (in the eighth grade) by completing the square or finding its roots.

Example 3. In solving problems related to the median of a triangle it is often helpful to use a construction consisting of extending the median a distance equal to the length of the median, thus constructing a parallelogram from the triangle. Using this construction, (a) we can prove the statements that a median constructed from a vertex of a triangle at which two unequal sides meet forms a greater angle with the smaller of these sides than with the larger side; that every median of a triangle is less than half the sum of the sides that it lies between; that the sum of the three medians of a triangle is less than its perimeter; (b) a formula for calculating the length of the median is developed; and (c) a triangle is constructed from two sides and a median produced between them.

Example 4. Having established a necessary and sufficient condition for a circle to be circumscribed about a convex quadrangle, the teacher also points out that if a circle is circumscribed about a convex quadrangle ABCD and we draw diagonals AC and BD of this quadrangle, we have divided each of its angles into two angles and obtained eight inscribed angles--based in pairs on a common arc--and consequently, the angles making up these pairs are equal: $\angle ABC = \angle ACD$; $\angle ACB = \angle ADB$; $\angle BAC = \angle BDC$; $\angle CBD = \angle CAD$.

This general remark can be used repeatedly in the proofs of theorems and the solutions of problems when a convex quadrangle about which a circle is circumscribed is given outright or can be incorporated as an auxiliary figure.

Example 5. In solving a great many problems on triangles, it must be remembered that if the sides AB and AC of triangle ABC are not equal (say, $AB < AC$), then when the altitude, the angle bisector, and the median of this triangle are dropped from vertex A, they are distributed in this order:

(a) smaller side, altitude, bisector, median, if angle B is acute;

(b) altitude, smaller side, bisector, median, if angle B is obtuse;

(c) altitude (smaller side), bisector, median, if angle B is right.

The pupils themselves come to this general conclusion after they have examined successively (under the teacher's guidance) the kinds of angles formed by sides AB and AC with the altitude, bisector, and median of the triangle, and into what parts these segments divide its base BC in the first and second cases mentioned above.

A set of exercises connected with the solution of this problem and containing (for the first case above) the following problems can be done by the pupils without difficulty:

(a) the altitude AD and the shorter side form a smaller angle than the altitude and the longer side; the base is divided into parts BD and DC such that $BD < DC$;

(b) the median AM forms a larger angle with the short side than with the long side;

(c) the angle bisector AL divides the base into parts BL and CL such that $BL < CL$.

Example 6. It is important to show that the points located at the center of a circle, the middle of a chord, and the middle of the two arcs cut by it, are all collinear.

In setting forth the theorem about the diameter of a circle that is perpendicular to a chord, it is useful to ask the pupils to formulate and prove all converse theorems.

Example 7. Various uses for the center of a segment must be combined in a single instruction; these uses are: (a) finding the locus of points equidistant from its ends, (b) finding the center of a circle circumscribed about a triangle, and (c) joining the center of a chord with the center of a circle (cf. Example 6).

Example 8. To prove that three points, A, B, C are collinear is identical to proving that the two half-lines BA and BC form a straight line.

Example 9. Each angle inscribed in a given circle completely determines the length of the arc on which it is based and the length of the chord which cuts off this arc.

3. Although the device indicated in section 2 helps the pupil to begin the proof of a hypothesis or the solution of a problem, it is quite insufficient, of course, for future progress toward the required goal. The question of the means that the pupil should apply to find the entire reasoning process on his own and to bring it successfully to an end is considerably more complex.

3.1. The famous French scholar J. Hadamard, in his Elementary Geometry [1], suggests the following rules that pupils can use in their independent search for ways to prove propositions:

First rule. "In the course of the proof, one must employ the hypothesis and, in most cases, the entire hypothesis"[1: 262]. This is a very important rule. It means that: (a) before proving a theorem one must establish its conditions precisely and with exhaustive thoroughness; and (b) when proving a theorem one must strive to use all points of its conditions without exception.

It is the content of a theorem's conditions that almost always leads the pupils to the proper way of proving it. The teacher wishing to lead the pupil in this direction nearly always says to him: "Turn back to the conditions of the theorem, consider what you know; can't you make use of this?" During this proof itself the difficulties, hesitations, and failures to follow through in reasoning can often be eliminated by considering the conditions that have not yet been utilized.

Second rule. "One must substitute definitions for the terms which they define"[1: 263]. We can explain this rule with the following examples:

(a) In the proof of a theorem on the bisector of an angle, the term "angle bisector" is replaced by the words "the half-line coming from the vertex of the angle and dividing it into two equal parts."

(b) In the proof of a theorem on the right angle, the term "right angle" is replaced by the words "the angle that is equal to its complement."

(c) Having defined the expression $\sqrt[n]{a}$, where $a > 0$ and $n =$ any natural number, by the relation $(\sqrt[n]{a})^n = a$, we use in the proof of each theorem establishing some property of the expression $\sqrt[n]{a}$ the definition which gives sense to this equation.

(d) The term "the distance from point A to line ℓ" is replaced by the words "the length of the perpendicular dropped from point A to line ℓ."

(e) The term "the center of the circle circumscribed about a given triangle (or inscribed in it)" is replaced by the words "the point that is equidistant from all vertices (sides) of the triangle."

This device (replacing a concept by its definition) can be successfully applied by the pupil from the moment he *approaches* the proof of a proposition or the solution of a problem; he should *discover* for himself, with the necessary clarity and thoroughness, the contents of the conditions (what is given, what is known?) and the conclusions (what are we trying to do?) of the proposition or the text of the problem.

For example, these replacements might be made:

The statement:	is transformed into the statement:
1. Point A belongs to a circle with its center O and its radius R.	1. Point A is located at a distance R from point O.
2. Points A and B belong to a circle with its center O.	2. Points A and B are at precisely the same distance from O.
3. In triangle ABC, segment AD is the altitude, bisector, and median.	3. In triangle ABC, $AD \perp BC$, $\angle BAD = \angle CAD$, $BD = CD$.

4. In triangle ABC, point 0 is the center of both the circumscribed and the inscribed circles.	4. In triangle ABC, $OA = OB = OC$, $OK = OL = OM$, where $OK \perp BC$, $OL \perp AC$, $OM \perp AB$.
5. Parallelogram ABCD is both a rectangle and a rhombus.	5. In parallelogram ABCD, angle A is a right angle, and the adjacent sides are equal.
6. KL is the mean line of triangle ABC.	6. In triangle ABC, the segment KL joins the mid-points K and L of two of its sides.
7. ABCD is an equilateral trapezoid.	7. In quadrilateral ABCD, $AD \parallel BC$, $AB = CD$.

These examples illustrate cases in which the "transformation" of a statement consists only in translating its contents into the language of mathematical symbols and bringing it closer to the form from which a precise description of its content may be derived.

In discovering the contents of the conditions, however, it is often possible to go further and replace a condition by one of its immediate consequences. For example:

The condition:	may be replaced by the condition:
8. Segment AB is longer than segment BC.	8. $AB = BC + CD$.
9. $\angle ABC > \angle DEF$.	9. $\angle ABC = \angle DEF + \angle KLM$.
10. Point P lies inside (outside) a circle with center 0 and radius R.	10. $OP < R$ $(OP > R)$.
11. Points A and B are located on the same side (on different sides) of line ℓ.	11. Segment AB does not intersect (intersects) line ℓ.

Similarly, the contents of a conclusion may be revealed by repeatedly replacing it with a statement from which it follows directly, and the proof of any relationship may be replaced by proof that a sufficient condition for the presence of this relation is satisfied.

3.2. In a lecture on "Directing Methods of Mathematical Research," delivered at the Second Congress of Mathematics Teachers in 1913, M. D. Osinskii suggested that the following general principles guide research in mathematics:

(a) The equality of segments and angles is proved from the equality of the triangles in which they are found.

(b) Second-degree equations are proved from the equality of triangles.

(c) An equality shows that its proof must be made.

(d) If the equality of two values cannot be proved directly, one or both of them must be replaced by an equivalent value and the equality of the latter proven.

(e) In the proof of theorems and the solution of problems, analogies with the proof of known theorems and with problems that have been solved must be used.

4. In conclusion, we present several examples in which we investigate how the pupil's ideas could arise and develop when proving theorems and solving problems. Here we shall strive of necessity to use the general instructions given above and to employ the general rules that have been formulated.

Example 10. Prove that a straight line joining the midpoints of the bases of a trapezoid passes through the point of intersection of its diagonals.

Proof. Let ABCD be given trapezoid with bases AD and BC, and let its diagonals AC and BD intersect at point M. We must prove that point M belongs to segment KL, which joins the midpoints K and L of BC and AD, respectively.

Thinking over the content of the conclusion, the pupil should transform it thus: "This means that we must prove that half-lines MK and ML lie on the same straight line." Arriving at this idea on the basis of the general instructions (Example 8), the pupil then attempts to prove that angles KMC and AML are vertical angles. To do this he examines triangles KMC and AML and establishes that $\angle KCM = \angle LAM$. But this is only a small step. For a proof of the equality of the third angles he must prove the similarity of triangles KCM and LAM.

But how can this be done? At this point he must reexamine the conditions of the theorem. He finds that he has not used the fact that ABCD is a trapezoid and that K and L are the midpoints of its bases. From the first fact he concludes that triangles BMC and AMD are similar; the second fact suggests that he should look at sides BC and AD. These sides are part of the similar triangles and, consequently, BC:AD = CM:AM. But then KC:AL = BC:AD, too. Therefore the triangles KCM and LAM are similar, so that $\angle KMC = \angle AML$. Consequently these angles are vertical angles, from which we conclude that half-lines MK and ML are the continuation of each other.

Example 11. Inscribe in a given circle a triangle in which the sum of two sides and the value of the angle opposite one of these sides are known.

Solution. Let us suppose that the problem has been solved, that is, that triangle ABC is inscribed in the given circle and that the sum of its sides (AC + BC) and its angle A, opposite side BC, are known.

If the pupil has thoroughly learned the general instructions in Example 9, he knows immediately that side BC of the triangle is completely determined by angle A and, therefore, that to find side AC it is enough to subtract side BC from the given sum.

A plan of construction will then become clear: (a) We inscribe the given angle in the given circle. (b) Its sides intersect the circle at points B and C, determining the segment BC; this segment is the side BC of the unknown triangle. (c) We find by construction the difference between the given sum (AC + BC) of the sides AC and BC and the segment BC; this difference is equal to the side AC of the unknown triangle. (d) From the end point C of the segment BC with a radius AC we describe an arc that intersects the circle at point A. The triangle ABC is the unknown triangle.

Example 12. Let ABC be an equilateral triangle inscribed in a circle, and let M be a point of the arc BC. Prove that segment MA equals MB + MC.

Solution. To show that segments MA, MB, MC are related by the equality MA = MB + MC, the pupils have no other means than to try to prove that some point will divide segment MA into two segments that are correspondingly equal to segments MB and MC. How can these segments be found? It is obvious that to find the unknown point one of the segments, say MC, must be laid out on segment MA.

As a result of this procedure, we find a point P dividing segment MA into segments MP and PA, with MP = MC.

The problem has been reduced to proving that PA is equal to MB. The equality of segments is most often established on the basis of the equality of triangles.

Of which triangles are segments PA and MB a part? Segment PA can be a part of triangle APC. Segment MB, however, is a part of triangles AMB and MBC. Both of these triangles have one side equal to one of the sides of triangle APC: AB = AC and BC = AC. But triangle MBC also has an angle equal to one of the angles of triangle APC: MBC = PAC. We shall therefore compare triangle APC with triangle MBC. A difficulty arises here. How can we go on? What have we not yet used?

Since triangles APC and MBC each have a pair of corresponding sides equal and a pair of corresponding angles equal, let us examine the remaining angles. We see easily that angle BMC = 120°. In triangle APC only angle APC can be equal to it, since angle ACP is less than angle ACB = 60°. But for angle APC to be 120°, it is sufficient that the adjacent angle at the base of the isosceles triangle, angle MPC, be equal to 60°, and consequently, that angle PMC at the vertex of this triangle also be equal to 60°.

From a drawing we see immediately that this is true. Consequently, triangles APC and MBC are equal, and AP = BM, which was to be proved.

Example 12 was presented not for class use but only as a demonstration of how to extricate oneself from difficult situations given a sufficiently large number of conditions.

Example 13. Prove the identity:

$$a(b + c)^2 + b(c + a)^2 + c(a + b)^2 - 4abc = (b + c)(c + a)(a + b).$$

First method. Following the instructions, the pupil will first attempt to transform the left-hand side of the equation into the right, for which he carries out the following transformations on the left-hand side of the equation:

(a) He removes the parentheses and multiplies out each term:

$ab^2 + 2abc + ac^2 + bc^2 + 2abc + a^2b + a^2c + 2abc + b^2c - 4abc;$

(b) he combines like terms:

$$ab^2 + ac^2 + bc^2 + a^2b + a^2c + b^2c + 2abc;$$

(c) following the instructions, he arranges this polynomial in decreasing degrees of a:

$$(b + c)a^2 + (b^2 + c^2 + 2bc)a + bc(b + c);$$

(d) he puts the factor (b + c) outside the brackets:

$$(b + c)[a^2 + (b + c)a + bc];$$

(e) he factors the polynomial that is inside the brackets

$$a^2 + ab + ac + bc = (a + b)(a + c).$$

Second method. The pupil, following another instruction, first tries to compare the coefficients of identical powers of a on the left and right sides of the given identity, for which he composes a table:

Power of a:	Coefficient of this power of a on the left and right sides of the equation:
a^2	$b + c = b + c$
a^1	$(b + c)^2 + 2bc + 2bc - 4bc = (b + c)^2$
a^0	$bc^2 + b^2c = (b + c)bc.$

Independent Execution of Exercises by the Pupils

Finally, since the pupil must necessarily work actively to solve problems in mathematics on his own both in class and at home, the teacher should try as often as possible to make him seek the solution of theoretical and practical problems independently. The pupil then would be applying his own energy to definite mental work without depending on the teacher's "nudging," although this does not mean that the pupil should be deprived of general guidance in his independent solution of the proposed problems. The teacher should begin utilizing this technique as soon as he starts teaching mathematics in the school.

The problem of developing the pupil's ability to do exercises in algebra independently can be presented and solved quite clearly. But the pupil's abilities to do exercises independently can be developed only if the system of exercises for each topic satisfies the following basic requirement. Each consecutive exercise must not repeat the preceding one exactly but *must be distinct from it in some basic principle*. The difficulty of these principles should be allowed to increase almost imperceptibly (for the pupils, of course), but each time they should force the pupil -- even if only for a very short time -- to think about the solution of the question, and to meditate on the applicability to this exercise of the method or device used in solving the preceding problems. Gradually the pupil should begin not only to reflect, but to delve more seriously into the essence of the question and to try drawing conclusions about the method or device best used in its solution.

Here the teacher should not miss any opportunity, when there is a possibility and when the need arises, to show the pupil the most natural and expedient -- or *rational* -- methods of solving an exercise or problem. This forces the teacher both to seek and establish these methods, and to communicate them in turn to the pupils as a conclusion and as a result requiring mastery on their part.

But it would be a mistake on the teacher's part if he omitted those exercises in the workbook that do not have a distinction in principle from their predecessors. Exercises of this type can be most helpful as "practice material" for work in class and at home.

Somewhat more complex is the problem of developing the pupils' ability to do exercises in geometry independently. Here it is significantly more difficult to attain a systematic order in the arrangement of exercises, an order which ensures the possibility of basing the solution of a given exercise on the experience of the preceding one or ones. It must be admitted, however, that those textbooks on geometry which are at the teacher's disposal do offer him a great deal of help in arranging exercises into a *system* satisfying the basic requirement stated above with regard to the exercises in algebra. The teacher must do this work if he wants to make the study of geometry feasible for the pupils while awakening their interest in and enthusiasm for the subject and their own creative abilities.

Laboratory Work for the Pupils

Until now we have understood "exercises that the pupils should do independently" to mean examples and problems from the appropriate problem books. Such tasks demand of the pupils only mental activity and concentration. But today teachers are also using the laboratory form of instruction, a form of instruction based on activating the pupils' senses, especially sight, touch, and muscular effort, as a stimulus to the pupils' activity. There are two fundamental types of laboratory work.

1. The first type has arisen as a result of the teachers' desire -- particularly in the fifth through seventh grades — to preface each general statement (conclusion, proof) with some experiment that will involve the pupil and convince him of the correctness of the concept being proved.

Even in the fifth grade, the pupils go as far as finding the formula for the circumference of a circle by the experimental method. Beginning in the sixth grade measurement is applied as a preparation to proving theorems, and the result is that each pupil — by means of inductive reasoning -- draws a general conclusion which, of course, must then be proven. In this way a number of the geometric theorems studied in the sixth and seventh grades are established intuitively before they are proven.

At the higher stages of geometry instruction (in the eighth grade), the deductive proof of a statement may still be prefaced with its inductive establishment based on measurement. This method is wholly admissible when the teacher considers it expedient for surmounting difficulties connected with the direct proof of a statement, or for ascertaining more concretely the essence of the statement itself.

For example, we may be convinced by measuring: (a) that the corresponding equality of the angles of two triangles implies the proportionality of their sides (a theorem which is false for polygons other than triangles); (b) of the theorem stating that the product of segments of chords passing through a given point within a circle retains a constant value, and so on.

Those experiments that are meant to improve the pupils' ability to visualize the solutions to geometric problems also belong to the first type of laboratory work. In them, imagined operations (such as placing figures on top of and beside one another, reflecting a figure about some axis of symmetry, rotating figures about a point, recutting figures) are replaced by real operations performed with the aid of appropriate figures cut from a material such as paper, wood, or tin.

It can hardly be doubted that these laboratory tasks, which bring the pupils' eyes and hands into action, promote mastery of the geometry course, which is otherwise difficult for some pupils to comprehend.

2. The second type of laboratory work is that directed at achieving an organic link between mathematics and industrial work. This type of laboratory work originated in the early practice of teaching a school mathematics course, gradually acquired a broader and deeper content in polytechnical education, and will now occupy quite a solid place in mathematics instruction in the schools.

If, as a result of the current school reform, mathematics instruction is to be closely linked with exposure to the methods provided by mathematics for solving practical problems, then the performance of the second type of laboratory work will represent an experiment in applying these mathematical methods to real-life situations, to actual productive work, and to reality in general.

This second type of laboratory work includes all those tasks which demand the complex application of methods of calculation, measurement, construction, and graphing; it also includes the performance of separate tasks in school shops and in the mathematics study room turned into a laboratory.

The school laboratory should be well equipped with all the necessary reference books and calculating instruments (abacus, calculating machine, slide rule); with millimetric, logarithmic, and semi-logarithmic paper; and later perhaps with simple electronic or mechanical computers.

The second type of laboratory work should include (a) graphic exercises, (b) measuring work, and (c) work on making models.

(a) Graphic exercises involve the construction of diagrams and graphs of functions, the graphic illustration of the solutions of equations and inequalities and systems of equations and inequalities, and the graphic solution and investigation of these equations, inequalities and systems.

The construction of diagrams, done in the fifth and sixth grades to promote the pupils' understanding of the idea of measurement and their conception of the idea of the functional relationship, is a first opportunity for pupils to use actively their mathematical knowledge.

As early as the sixth grade, pupils go from the construction of diagrams to the construction of graphs of functions. The major problems associated with this idea, organically connected with the problem of constructing a mathematics course on the idea of the function, are solved gradually, in a definite sequence; and the subject is examined with great thoroughness and depth in the upper grades as a means of building up related material.

The pupils first acquire skill in constructing graphs by plotting points. At this stage the pupils' activity is manifested in plotting points on a graph from given coordinates; they begin using the class blackboard covered with a coordinate grid, then paper ruled off in squares in their ("arithmetic") notebooks. Gradually, an increasing independence and activity in performing assignments on constructing graphs of functions is demanded.

Beginning in the eighth grade (and indeed, to some extent in the seventh grade, as well) the pupils construct graphs of functions after a preliminary analytic investigation of their properties. At this stage the construction of graphs of functions unifies the pupil's mental and physical activity in a combination of theory and practice. These exercises are usually of great interest to the pupils, especially if the functions illustrate a law governing some natural phenomenon which they have studied in science (physics, chemistry, technology, or biology).

Exercises in constructing graphs of functions have so much value for the study of mathematics in the school and in institutions of higher education, as well as for the pupils' future practical work, that the teacher should not limit himself to the types of functions indicated in the curriculum, but should freely use others which are similar in construction to those being studied. Thus, having acquainted the pupils with the method of transforming the graph of the function $y = x^2$ into graphs of the functions

$$y = x^2 + b, \quad y = (x-a)^2, \quad y = kx^2, \quad y = k(x-a)^2,$$

the teacher may suggest that the pupils independently apply these transformations (translations, expansions, and compressions) to the graphs of other functions.

Finally, the construction of graphs may help in investigating equations and systems of equations and may provide graphic solutions to them. This method is especially valuable when the analytic solution of an equation or a system of equations is particularly difficult. The graphic investigation of an equation or an inequality containing a parameter is very interesting and instructive when that investigation consists in constructing a graph representing the relationship of the parameter to the unknown equation or inequality; this work proceeds in a non-standard way.

(b) Measuring work, when properly organized (done partly in the classroom, but mainly outside it), constitutes the active type of task that should aid in the pupils' polytechnical training and in communicating useful practical information to them.

(c) The kind of modelling that involves the pupils in the actual construction of models develops elementary skills in construction and handicraft work in general (in work with wire, cardboard, glass, tin), and develops the pupils' creative abilities -- often forcing them to construct models of complex geometric figures resulting from the proof of theorems and the solution of problems. Construction of models illustrating the change of various elements of a figure during the course of the change of some parameter opens an especially wide field for the pupils' creative activity.

The Use of a System of Questions Designed to Give Depth to the Pupils' Knowledge and Development

A prepared system of questions -- applicable during drill on a newly studied theory, as well as in reviewing it and in checking the pupils' knowledge — may be used to a significant extent not only to make the pupils participate more actively, but also to broaden their mathematical knowledge. The concepts and facts (relations, conjunctions) dealt with in these questions must be subjected to thorough examination; their essence should be explained so that the ability to answer the questions correctly will indicate how well the pupils' degree of logical thinking and creative abilities have developed.

This idea (of the need to bring about the pupils' conscious mastery of the study material by developing their ability to solve theoretical questions independently) has led to the appearance of a number of articles and books to assist the teacher in his work. We refer the teacher to them.

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INDEPENDENT WORK FOR PUPILS IN ARITHMETIC LESSONS IN THE ELEMENTARY GRADES*

M. I. Moro

Problems and Methodology of Investigation

The Twenty-second Congress of the Communist Party of the Soviet Union presented our schools with the problem of the comprehensive harmonious development of the child's personality, of imparting the qualities of future builder and member of a communist society. In the decrees of the party and the government, it was recommended that instruction in the school be built on the basis of the principle of a close relationship with life, and that all educational work should be directed toward a development of children's cognitive skills and toward instilling in children independence, activity, initiative, and creative principles. From this recommendation the following requirement can be inferred, which, in our view, is a most important one. In the instruction process conditions should be created that provide opportunity for the children's systematic exercise in the independent application of previously acquired knowledge to the solution of various educational and practical problems, as well as in the independent acquisition of new knowledge.

In the course of systematically conducted (on which children work independently) tasks organized for all lessons, the children should be instilled with a feeling of duty and responsibility for an assigned task, and with the persistence and tenacity essential for overcoming the difficulties that arise in solving a problem in school or in work. There is no doubt that without using independent work for children as one way of organizing educational activities, it is impossible to solve the problems that have been posed.

Questions of the content and methodology of directing the pupils' independent work therefore presently assume a special timeliness and acuteness. Many works by Soviet didacticians and methodologists contain a number of useful hints and recommendations on these questions, and the literature that illuminates the practical experience of the best teachers has been enriched, but until now the questions that arise in connection with the problem of developing the pupils' independence have not yet been completely solved.

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*Of the Institute of General and Polytechnical Education, Academy of Pedagogical Sciences of the RSFSR. Published in the series _The Teacher's Pedagogical Library_, Moscow, Publishing House of the Academy of Pedagogical Sciences of the RSFSR, 1963. Translated by Lydia Hooke.

In the present manual we consider the place of independent work for pupils in arithmetic lessons, its aspects and forms at different stages of instruction, features of organization, and the system and methods of its execution. The starting point in the investigation was a study of all those aspects and concrete forms of independent work for pupils that are used in the practice of arithmetic instruction in the lower grades. In connection with this study, one of our chief methods of research was the observation of the work of superior teachers, and an analysis of, and generalization from, their experience. For this we used not only data from our own observations but also facts that have been elucidated in the methodological literature.

In the academic year 1959-60 we studied the organization of pupils' independent work in arithmetic lessons, attending the lessons of superior teachers of grades 1-4 in schools in the Moscow, Volgograd, Yaroslav, and Orlov provinces, and in 1961-62 in the Moscow and Lipetsk provinces. Altogether, more than 200 lessons were analyzed. From the records of the lessons, a rigorous study was made of the time spent on various activities. The assembled data characterizes the content and methodology of the execution of practical work in arithmetic lessons, and allows one to judge the place set aside for it in practice, the average duration of various aspects of assignments, the relationship between the work performed independently by the pupils in class and that conducted with the teacher's direct participation and help. The appropriate data will be examined below.

Since we did not limit ourselves to observing lessons, in each school we became familiar, in the methodological centers, with those visual aids and didactic materials that are used in arithmetic instruction, and we ascertained, through conversations with teachers, the methodology of their application. Moreover, we studied the class registers and pupil notebooks, trying, with the teachers' assistance, to understand the sort of independent tasks in arithmetic cited in them,the content, how they were organized, how checking and appraisal is accomplished, and the like. We attended individual classes of superior teachers in the elementary grades, but we had no opportunity to become acquainted with the general system of their work.

Systematic observations of the arithmetic instruction of children in school No. 315 in Moscow (teachers A. M. Logacheva, Z. V. Kozhokina, L. E. Zaikina, T. V. Titova, M. A. Korosteleva) were made with these objectives in mind. In the first stage of the investigation (1957-1959) the observation was limited to the work of the best teachers in the school. During this observation, we introduced almost nothing that was new in principle into the organization of independent assignments in class. In the next stage (1959-1961) the character of our work in school No. 315 changed--first in the first grade, then in the second grade (teachers L. E. Zaikina and T. V. Titova). Experimental instruction was organized that was constructed in light of our study and generalization of the work experience of the best teachers and our analysis of the literature and of the educational problems to be faced by the teacher at each stage of arithmetic instruction. In the organization of experimental instruction in grade 1, we also considered the results of experimental work that we had previously conducted in the first grades of the same school. This work was devoted to the study of the level of arithmetic preparation of seven-year-olds who enter first grade, to the special features of working with them at the first stage of their school instruction [19] and to the elaboration of a system of visual aids and didactic materials in arithmetic for grade 1 (after testing the system of such didactic materials in the course of experimental instruction, we described them in our pamphlet) [11].

Parallel with the experimental instruction in class, we systematically conducted experimental classes with individual pupils (as a rule, average and poorly prepared) and with small groups of pupils. The purpose of these classes was basically a preliminary check of the various aspects of assignments that we had outlined for subsequent lessons. In the course of these assignments, we studied the difficulties that the children experienced in fulfilling an assigned task and outlined ways of forestalling these difficulties. The latter consisted of modifying the formulation of instruction; breaking down the assignment into smaller

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parts; and devising some additional preliminary exercises; special visual aids, didactic material, and the like. Each change introduced into the content and methodology of the outlined work was tried out afresh during classes with other pupils.

After having defined the tasks as a result of this preliminary selection, we then tried the system with the entire class participating. During the trial, further perfection and specification of the content, organization, and procedure of the tasks was continued.

We conducted some lessons in which new assignments were given in order to have an opportunity to introduce necessary amendments immediately in class. In a number of cases we had to determine the relative value of various aspects of assignments, or of various methodological devices that could be used in conducting the children's independent work. To clarify their advantages and defects in comparison with one another, we subjected them to an experimental check in two parallel classes; using a different methodological approach in each (or a different assignment).

During the school year 1961-62, work in the same direction was conducted in grades 3 and 4, in which we observed the arithmetic instruction of children by superior teachers. In one experiment questions related to features of the organization of independent work were investigated. Principal attention was directed to determining the possibility of using independent work at the stage in which the pupils are familiarized with new educational material. This question was examined in the material for the topic "Million" in grade 3. In two experimental classes the study of this topic was conducted differently. In one class the basis for familiarization with the new material was the teacher's explanation, and in the other it was the pupils' independent work. This gave us an opportunity to compare both methods and to answer the problem that was posed. The system of children's independent work in arithmetic instruction was elaborated only in material for grades 1 and 2 (the creation of such a system for grades 3 and 4 is a matter for the future). The system outlined was tested in the work experience program of 25 teachers in Moscow schools who were students in the annual courses in improving one's qualification at the Moscow advanced training institute for teachers in 1960-61. The present work is by nature a methodological aid, which was intended to give concrete help to teachers in organizing and conducting independent work for children in arithmetic lessons.

However, before moving on to the practical part of the work, we should examine some general theoretical questions related to a definition of the concept of "pupils' independent work" and to classify various aspects of such work used in arithmetic instruction in the elementary grades. Confusion in the solution of these questions is, we feel, one of the obstacles to introducing, into the broad practice of instruction, methods that would create the most favorable conditions for developing pupil's independence, initiative, and creativity.

Fundamental Aspects of the Pupils' Independent Work

Definition of the Concept "Pupils' Independent Work"

In our recent pedagogical literature a debate has developed over the meaning of "independent work." Many authors concentrate on an analysis of the essential features of children's activity in performing various types of scholastic and practical tasks. Comparing the diverse forms of assignments used by teachers in their school practice, these authors have shown that the pupils' activity often amounts to imitation, or the precise execution of the teacher's instructions. At first glance, such work demands no independent thought of the child--no initiative or independence in stating the problem or seeking a method for its solution. Because of this, some authors are not inclined to include imitation in the ranks of independent work. For example, R. B. Sroda writes, "By the pupils' independent work we understand work in which they manifest a maximum of activity, creativity, independent judgment and initiative" [21: 7]. The contrast between "imitative," "careful" activity on one hand, and "independent" activity on the other, may be found in statements by E. Ya. Golant, R. G. Lemberg, and others.

In the final analysis, such a contrastive distinction leads to an extraordinarily narrow understanding of independent work, which would not include such activity as solving examples of the problem types familiar to the children that are analogous to problems they solved earlier with the guidance and aid of the teacher. Under these conditions the demand for raising the proportion of the pupils' independent work during the educative process would have to be understood only as a demand for frequent utilization of so-called creative tasks in instructional practice. Even though the role of such work, which opens the greatest possibilities for the children to evince their independence of thought, initiative, and creativity, is great, one must admit that by such an approach important possibilities of raising the efficacy of educative tasks (including training tasks) could be overlooked by investigators studying these questions, as well as by practicing teachers. Nevertheless, observations of arithmetic instruction in the elementary grades show that questions of the content, organization, and method of carrying out these tasks (extremely important in teaching arithmetic) still remain far from clear.

Training assignments are important not only for developing appropriate arithmetical skills, but also for getting children to master a whole series of facts, abilities, and skills of independent work without which one cannot imagine any kind of creative activity. To clarify our understanding of pupils' independent work, let us examine a concrete example from instructional practice that was evaluated in contradictory ways in two recent works dedicated to this subject. R. G. Lemberg [7] used this example as an illustration of the incorrect understanding of the term "scholastic independence" of pupils, and B. P. Esipov [5] used it as one of the forms of the child's independent work. The example concerns that period of instruction during which children master the ability to write letters (and numbers). Lemberg stated: "For a long time Peter could not write capital letters; he had to be continually helped, but now he writes independently. 'Here he has drawn the letter B exactly as on the blackboard,' says the teacher". Then the rhetorical question followed whether the term "independently" was used correctly which is followed by the categorical answer, "of course not," and by the explanation: "The little boy wrote skillfully, not independently." Lemberg then produced, for comparison, an example of the correct (in her opinion) usage of the term "independently". The pupil reads aloud and

> his intonation and gestures do not copy the teacher's manner, but express his independent relation to the text, his feelings, his understanding. . . . Of course, the concept 'independent operation' should assume not so much the pupils' independence from guidance as their introduction of something personal into the work.

By this criterion Lemberg distinguished "performance" activity, which consists of "listening to the teacher, doing his instructions and assignments" from independent activity, which consists of a "certain contraposition to it."

If this division is thought through, the question arises whether the pupil may bring something "personal" into the work even when he works an assignment of the teacher's which demands that some given instruction be followed. Say, for example, that the teacher explains and shows first-graders how to write the numeral 5, gives a model of the writing of this figure, and asks them to write it five times in their notebooks. B. P. Esipov was apparently correct when he wrote that "if work on such assignments is organized so that the pupil doing it will consciously strive to best achieve the aim, i.e., strive for the best quality shown by the model, this work may be called independent work" [5: 5]. In fact, when doing such an assignment, the child should evince purposefulness; determination in overcoming difficulties; an ability to make comparisons independently; and a capacity to approach his own work critically, to evaluate its results, and (on the basis of this critical examination) direct his efforts to eliminating errors he made in previous stages of his work. For him, this is connected not only with muscular sensations and the need to concentrate his attention, to strain his will, to impress his consciousness with a responsibility for the quality of his work, but also with his active thinking.

At definite stages of instruction even the fulfillment of assignments, seemingly based on simple imitation and on the precise following of instructions, in fact depends upon the children's showing a certain amount of independence and demands of them active conscious participation. Such fulfillment of assignments therefore can rightfully be related to the category of pupils' independent work. However, owing to the pupils' development during the instruction process and to their mastery of appropriate facts, knowledge, and skills, the nature of the children's activity in doing the same assignment changes so that it ceases to involve evincing of cognitive activity. As a rule, fulfillment of the teacher's assignment without direct assistance on anyone's part demands that the children show independence (even though in an extremely limited amount).

Let us turn again to an example from teaching practice. Teachers are well acquainted with situations in which the child, doing his homework assignment or taking a test in class, cannot cope with the assignment just because he does not feel the teacher's direct support, which he has grown accustomed to under the conditions of collective work in class. Here it is enough for the teacher to come up, stand awhile alongside such a pupil, and nod his head to signify that all is correct. Then new strength seems to flow into the child, and he confidently continues his work.

Lack of self-confidence, indecisiveness, the need for constant minor protection from somebody else, the lack of elementary independence (which makes it impossible to apply one's knowledge, abilities, and skills even when doing a familiar type of assignment)--all this is unfortunately still characteristic of pupils in the lower grades.

Therefore there can be great value in organizing the children's study activity so that they work alone, without the teacher's direct participation, on various assignments. This idea was expressed in the definition of independent work given in R. M. Mikel'son's book: "By independent work we mean the pupils' completion of assignments without any assistance, but under the teacher's observance" [9].

In many articles that appeared after Mikel'son's book this aspect of children's independent work--the absence of anyone's assistance when doing an assignment--was emphasized. Some authors, however, perceived in such a treatment of children's independent work a depreciation of the teacher's guiding role in the educational process. Criticizing Mikel'son's definition, they pointed out that all the pupils' work, including independent work, needs constant control and guidance from the teacher and that in conducting the children's independent work, the teacher is obliged to render them necessary assistance.

Thus, for example, in A Reference Book for the Elementary-School Teacher, we read:

> The children's independent work in class should not be a form of instruction which in any way reduces the teacher's role. On the contrary, developing the children's ability to work independently is possible only when the teacher systematically guides the children's independent execution of assignments, explaining devices of work, observing the process of the execution of the assignment, correcting mistakes and helping the children overcome difficulties [8: 159].

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In this and similar arguments there is a confusion of two different things--the question of the necessity of the teacher's guidance and the question of rendering direct assistance to the children during the work process.

The fundamental characteristic of children's independent work is, in our opinion, the fact that during its completion, the pupil is deprived for some time of his accustomed guardianship from the teacher and remains alone with the problem before him whose solution involves definite (perhaps even significant) but surmountable difficulties. In working on such an assignment, the pupil must put his strength to the task and--relying on his own knowledge, abilities and skills; keenness of observation; quickness of wit; and sometimes ingenuity—find a way to solve the problem and complete the solution.

It is quite natural that mistakes may arise in the pupil's independent work; that one mistake may lead to another; and that an incorrect answer will be obtained in the result. Does this mean, however, that the teacher must halt the pupil's work at the point where a difficulty occurs when a real threat of a mistake in the solution has appeared? Must measures be taken so the possibility of an error's arising is always foreseen by the teacher so the teacher, interrupting the pupil's path of argumentation, would eliminate the difficulty and redirect the pupil's thought and actions along the right channel? Is it not more correct to give the pupil a chance to realize his error in the solution, to attempt to independently to find his mistake and correct it? It seems that the last-mentioned pedagogical method is significantly more expedient whenever the pupil is well enough prepared for the corresponding work.

The teacher is forced to interfere in the pupil's work—to give him direct aid in his completion of an assignment--only if the proposed assignment is not at the appropriate level of the pupil's attention. In this case the teacher takes steps to eliminate whatever difficulties there are (either by simplifying the assignment or by rendering the pupil direct assistance), but by doing so he deprives the pupil's work of its genuine independence. In our view, the pupil's genuinely independent work is only that work which is done without direct assistance from the teacher—without the teacher's direct participation in the work at the time it is being done. This does not exclude, but presupposes, the teacher's guidance.

The definition given the concept of pupils' independent work in the book by Esipov cited above gives more insight into the essence of the matter. This definition is presented in its entirety:

> The pupils' independent work embraced by the educational process is work done without the direct participation of the teacher, but by his assignment, in a time period presented especially for this purpose. Here the pupils consciously strive to achieve the aim presented in the task, using their capacities and expressing in one form or another the result of mental or physical (or both simultaneously) operations [5: 34].

It must be further stressed that independent work by necessity would be connected with the children's conscious activity, that the efforts which the children make to achieve the proposed aims should be directed toward surmounting not just any difficulties (any work generally involves the need to make certain efforts toward its completion), but the difficulties connected with the solution of one or another cognitive problem.

And so, by pupils' independent work we shall understand a form of organization of the children's cognitive activity in which they consciously and actively strive to attain the proposed aim, overcoming difficulties they meet on their path without anyone else's help in the course of completing the work. If in his observation the teacher is convinced that an assignment is unintelligible to some pupils, he should give them the necessary assistance or replace the assignment with an easier one. In the future he should prepare these children for completely independent solution of a similar assignment. In conducting the children's independent work it sometimes appears that the assignment given by the teacher needs additional explanation. If so, the teacher interrupts the children's work and gives additional instructions which make the work more intelligible.

By this understanding of independent work it becomes clear that a more frequent use of such types of work broadens the possibility of exercising the children in independent application of their knowledge, abilities, and skills and in independent mastery of new facts, which is the main and fully necessary condition for the development of pupils' independence as a personality trait.

Classification of the Basic Aspects of Pupils' Independent Work in Arithmetic Lessons

In their practice the best teachers use various devices to conduct the pupils' independent work, utilizing diverse forms of exercises. In teachers' works one may find numerous examples of assignments used in arithmetic lessons, valuable ideas, and apt observations indicating which requirements should be filled by such assignments and which difficulties are encountered by the children in doing these assignments (see, for example, [10, 12, 20]). The analysis and generalization of all this material is, of course, still far from completion. There is still no complete agreement on the proper classification of various types of independent work.

Without involving ourselves deeply in the juxtaposition of various points of view, we shall formulate here only those conclusions we have drawn on the basis of a critical examination of the literature, an analysis of the work experience of the best teachers, and the observations of pupils in their completion of diversified exercises as independent work.

From all this data we may propose the following classification of various types of independent work.

1. Using as a criterion the pedagogical aim pursued in conducting independent tasks, we may divide them into basic groups—instructive tasks and checking tasks. *Instructive tasks* can be subdivided into: (a) tasks preparing the children to perceive new study material; (b) tasks in acquiring new knowledge; (c) tasks directed toward expanding and deepening the acquired knowledge; and (d) tasks of a training nature whose aim is to secure earlier-mastered knowledge, abilities and skills. *Checking tasks* can be subdivided into: (a) tests whose aim is to ascertain and evaluate the pupils' knowledge; and (b) test work having no ascertaining nature, conducted by the teacher in order to specify the level of the children's preparation and their possibilities. Such specification is essential for a correct definition of the content and methodology of future instruction (the work done on the latter tests cannot be evaluated, for it is conducted, for example, at the beginning of the year in order to establish what part of previously studied material has been forgotten

by the children over the summer vacation, or at the end of the school year in order to review the material studied during the year and so on).

2. Independent tasks may also be subdivided into types according to the nature of the activity they demand of the pupils: (a) those based mainly on imitation, or reproduction by the pupils, of the teacher's actions and his arguments; (b) those which demand that the children independently apply knowledge, abilities, and skills acquired earlier under the teacher's guidance, under conditions analogous to the conditions in which they were formed; (c) those tasks which demand the same application as in (b) but under conditions more or less distinct from those under which the knowledge, abilities, and skills used by the children in doing the assignment were formed; and (d) so-called creative tasks which demand that the children show independence in posing the question and in seeking ways of solution--independently making the essential observations, obtaining a result, and selecting the material needed to compose a problem.

3. Independent tasks may be subdivided according to the curriculum material on which they are performed. In arithmetic instruction in the elementary grades, this feature may be used to distinguish work directed toward forming basic arithmetical concepts and work connected with teaching the solution of arithmetic problems, arithmetic examples, practical work (measuring, etc.).

One might list several other criteria by which independent tasks may be distinguished but which have lesser significance. Thus, such tasks may be distinguished, according to their form of organization in the lesson, as general classwork (in which the whole class works on a single assignment), group work (in which separate groups of pupils work simultaneously on different assignments), and individual work (in which each pupil receives an individual assignment from the teacher). Many methodologists also subdivide pupils' assigned work into fully independent and semi-independent work. By semi-independent work we usually understand work partly conducted with the direct participation of the teacher. It seems expedient to refrain from using the term "pupils' semi-independent work" with this meaning, since it conceals the distinction between non-independent and independent work.

It is more correct, from our point of view, to consider as separate that part of the work done with the teacher's assistance (and is therefore not independent), and that part of the work characterized by the features essential for independent work. Then the practicing teacher and the investigator studying the features of the children's independent work will always concentrate on this part of the work, and will consider its other part (conducted by the children with the teacher's assistance) only as preparing the children for independent execution of a more specific assignment.

Let us return to the classification of independent work given above. It is clear that a single task may be considered of different types if it is evaluated from different points of view. Such a versatile approach should help the teacher diversify the types of independent tasks and note their system. In addition it must be remembered that all the types and forms of tasks mentioned above (even within a single group) rarely appear in complete isolation in teaching practice. More often an assignment for independent work embodies more than one type of work, sometimes falling into a single classification group. Some assignments, however, seem to be transitional from one type to another.

This transitional nature is demonstrated through an example of children's solution of a familiar problem. Such independent work may be considered, as a rule, among the tasks directed toward broadening and deepening earlier-acquired knowledge (insofar as the independent solution of each new problem, even of a familiar type, furthers the children's deeper realization of the characteristics of problems of a given type and of their solution method),and among the tasks intended to prepare children to perceive new material (e.g., when the teacher presents a simple problem for independent solution before he begins to examine, with the children, a problem of a more complex type including, in particular, this previous, familiar problem). This work may test the pupils, but at the same time it will be instructive, since the children learn something in the course of its completion, perfecting and strengthening their previously obtained knowledge, abilities, and skills.

Thus if the division of independent work is examined even by the first feature mentioned--it is evident that this division has a relative character. It may be a matter not of a strict differentiation of various types of independent work according to a selected criterion, as is done in any scientific classification, but rather of isolating the essential characteristics of the work (again according to an isolated feature) in every concrete case of its application in practice.

With an approach to the types of independent work like that mentioned above, the pupils should not be surprised if a single task, at the same stage of instruction and in identical circumstances, is appraised for example, at one time as instructional and at other times as verificational, because in conducting it, the teacher may have pursued one goal in one case and another in the other. Despite the condition of a given division of independent tasks into types, the division has practical value in that it helps specify the goal and place of each such task in the instruction process and notes the corresponding methodical devices. In particular, depending on the teacher's aim when he proposes a familiar problem to the children for independent solution, in the case examined, this work will be conducted in various ways in class and the pupils will be variously prepared to do the assignment in class. The nature of the teacher's guidance of the children's work is determined, in a given case, by the exact aim of the work.

What has been said above may be extended to those subdivisions which relate to groups of tasks isolated according to other features. The solution of any arithmetical story problem involves the performance of various calculations. Therefore, the independent work constituting the solution of a problem may also be viewed as an exercise in calculation. If so the teacher, in thinking over the content and character of the instruction to be given the children before the corresponding work is undertaken, should take into account those difficulties that children may encounter in connection with planning the solution of a problem, and those that are connected with calculation or renaming. Having noticed, for instance, essential difficulties in a problem he intends to use, the teacher can consciously simplify the work before presenting the problem so as to make the assignment within the power of the pupils and also to allow the pupils to concentrate on that aspect of the problem which is (from the teacher's point of view) most important at the time. Another time, the teacher may consider it expedient to keep the content of the problem intact and to organize the instruction to assist the children in overcoming difficulties of a different type, etc. But in all cases it is important that the teacher take into account the aim of his contemplated assignment and to which of the types listed above it belongs.

Thus, we see the practical value of the above classification of various types of independent work in arithmetic instruction in the elementary grades. This classification should help the teacher in his analysis and critical approach to the assignments he intends for independent work and to the instruction and other forms of guidance that he proposes to use in conducting these tasks. A wide knowledge of all types of work should help the teacher in selecting a system of such assignments and in facilitating their diversity and gradual increase in difficulty.

It is evident that if one does not clearly imagine all possible types of independent work for children and if one does not group this work according to definite features, it will be impossible to begin to develop a system of such work. This problem demands, moreover, a very careful study of the characteristic aspects of content, the organization and method of executing independent tasks of each type, and a definition of their place of independent tasks in the pedagogical process.

The Place of Independent Work in Arithmetic Lessons

The Proportion of Children's Independent Work in Each Lesson

Our method of observation of arithmetic lessons included a precise indication of the length of time of its separate parts, which made it possible to consider all types of independent work used by the teachers and the time allowed for each. The data from observations during the 1959-60 school year are given in Table 1. Lessons from schools in the Volgograd, Orlov, and Yaroslavl' Regions were analyzed. The analysis of lessons from the Moscow Region was limited to 14 lessons of various Moscow teachers and 6 lessons observed in Perovo. We purposely omitted material of the work of teachers of the base schools of the Academy of Pedagogical Sciences of the RSFSR, where our experimental work was conducted at this time. In Table 2 the data characterizing the organization of the pupils' work in schools of the Lipetsk Region are presented.

The first thing of importance is that of the 100 lessons analyzed (see Table 1), independent work for pupils was absent from only 10. This indicates that the concept of a good lesson is already firmly connected in the minds of the best teachers with the necessary organization of such work. Further, the teachers used independent work 120 times during only 90 lessons. Thus, in some lessons independent work for pupils was given more than once. In addition to the data given in Table 1, the distribution of independent work by lessons is given as follows: no independent work--10 lessons; one independent tasks--64 lessons; two independent tasks--21 lessons; three independent tasks--five lessons.

These data are now compared with the data characterizing the lessons of teachers of the Lipetsk Region contained in Table 2. In the lessons represented by Table 2, independent work for pupils was used significantly more often than in lessons represented by Table 1. For Table 1, there were 120 cases of independent work in 100 lessons and, for Table 2, there were 57 cases of independent work in 12 lessons. The pupils' independent work was not observed at ten lessons, was observed at 64 lessons one time each, and was observed at only five lessons three times each, for the cities in Table 1. There was not a single lesson then, in which the teacher did not conduct at least three independent tasks, and during certain lessons (see, for example, the second-grade lesson, conducted by Sushkova, in School No. 19 of Lipetsk) their number reached nine.

Those 12 arithmetic lessons in the elementary grades which we observed in schools of the Liptesk Region were distributed as follows: no independent work--none; three independent tasks--(at least) one lesson; four independent tasks--(at least) seven lessons; five-six independent tasks --(at least) three lessons; more than six independent tasks--one lesson.

If the data of the tables characterizing the proportion of independent tasks of the Lipetsk Region teachers with that of teachers of other regions are compared, there are very meaningful differences. Actually, it is clear from Table 1 that at 100 of the lessons observed, the children worked independently for approximately 1000 minutes-- 22.2 percent of the total school time. From the Lipetsk materials this percentage is 64.7. It is clear, then, that Lipetsk Region teachers, having organized

TABLE 1

THE USE OF INDEPENDENT WORK IN ARITHMETIC LESSONS IN GRADES 1-4.

City [90: 10]**	Type of Independent Work: No. of Cases and Times*					
	Copying from Blackboard	Solutions of Examples	Partially Independent Solution of Problems	Independent Solution of Problems	Independent Work in Learning New Material	Total
Volgograd [25: 1]	6:(15,4,21,7,10,13):70	16:(8,5,4,6,7,3,3,3,4,6,11,4,10,7,6,4):91	6:(25,7,14,12,15,8):81	3:(14,6,7):27	1:(4):4	32:273
Novo-Annensk (Volgograd Region) [12: 2]	4:(10,13,7,4):34	8:(11,17,3,7,12,8,10,6):74	2:(20,16):36	3:(10,6,6):22	..	17:166
Orel [16: 2]	3:(4,6,10):20	9:(7,4,3,8,6,5,6,4,7):50	2:(10,12):22	3:(5,7,10):22	2:(6,4):10	19:124
Perovo (Moscow Region) [6: 0]	4:(5,8,6,4):23	6:(15,11,12,8,13,6):75	1:(11):11	1:(8):8	..	12:117
Yaroslavl' [7: 3]	2:(10,6):16	4:(8,10,12,6):36	2:(10,7):17	2:(8,6):14	1:(10):10	11:93
Gavrilov-Yam [10: 2]	2:(6,4):10	6:(8,10,13,6,14,7):58	2:(10,12):22	2:(10,7):17	..	22:107
Moscow [14: 0]	2:(7,4):11	6:(10,12,6,8,12,5):53	3:(8,6,12):26	4:(12,12,7,10):41	2:(8,4):12	17:143
Total	23:184	55:437	18:215	18:151	6:36	120:1003

*x: (a,b,...):y means x cases; the first of a minutes duration; the second of b duration; etc. for a total of y minutes

**[a:b] means a lessons in which independent work was used and b lessons in which independent work was not used, total and by city.

TABLE 2

THE USE OF INDEPENDENT WORK IN ARITHMETIC LESSONS IN GRADES 1-4: SCHOOLS OF LIPETSK REGION

		Type of Independent Work: Time in Minutes										
School	Grade	Copying from Black-Board	Comment on Solution of Examples	Solution of Examples	Other Types of Work on Examples	Partially Independent Solution of Examples	Comments on Solutions of Examples	Independent Solution of Problems	Other Types of Work on Problems	Practical Tasks	Total Time Devoted to Independent Work	Number of Tasks of Different Types Conducted at the Lesson
October	1	1	..	10;3	..	5	..	..	..	..	19	4
	2	..	7	9	..	..	6	8	..	..	30	4
	3	..	5	..	..	3	4	10	..	..	22	4
	3	..	4	..	..	5;2	..	8	..	..	19	4
	3	..	..	7	..	2;10	..	..	..	..	19	3
	4	..	..	7	..	16	..	14	..	3	40	4
Lipetsk No. 19	1	..	5	..	3	3	5	10;8	..	..	34	6
	2	..	5	..	2;4	10	3	8;6	..	..	24	4
	2	..	7	..	2	11	..	4	2;2	..	42	9
	4	..	..	11	..	7	4	7	..	7;2	38	6
Lipetsk No. 1	2	..	..	6	3	6	7	9	..	..	31	5
	3	..	5	4	12	8	..	..	..	..	29	4
Total for 12 Lessons		1/1	8/38	7/53	6/18	13/92	7/37	11/92	2/4	3/12	347	57

Note: In the columns having two numbers in the Total, the first number designates the number of cases involving use of independent work of the given type and the second number designates the total time (in minutes) devoted to tasks of this type for a total of 12 lessons.

the problem of increasing the efficacy of arithmetic lessons, followed the path of increasing the proportion of independent work per lesson at the expense of other types of activity. They did this, significantly, not by increasing the duration of each task, but by more frequent inclusion of independent work in the process of the lesson. Is this path correct? From our observations of the teaching process in experimental classes, we can affirm that this tendency, realized in the practice of the teachers in the Lipetsk Region, satisfactorily answers the problem of raising the efficacy of instruction. In analyzing the problems of each individual lesson, selecting exercise materials corresponding to these problems, and determining the methods of carrying them out, we were convinced each time that a lesson cannot be of full value without including pupils' independent work.

For the duration of the experimental instruction in grades 1-3 of School No. 315 in Moscow, there was not one lesson in which independent work for pupils did not have a place or was not necessary. Striving to increase the pupils' activity at each stage of the lesson, we usually conducted independent work not once, but two or three times, and at some lessons even four times.

However, both the frequency of independent work at lessons and the time devoted to it depend on the nature of the curriculum material, and change from one topic to another. Even the duration of individual tasks changes. Although in the beginning of the first year of instruction the problems for independent work are most often calculated at three or four minutes (sometimes less) and a maximum of ten or twelve minutes, by the end of the year (for example, at review lessons) it is possible to devote up to 20 or 25 minutes to such tasks. The most often encountered tasks, however, are of 5-7 minutes duration. In later grades such regularity was not observed. Beginning with the middle of the first year of instruction and in subsequent grades, short tasks (5-7 minutes) and tasks of a longer duration are both used with equal right. The amplitude of the oscillations in this respect is very great (2-3 to 35-40 minutes).

The same applies to the frequency with which this form of organizing class activities is used. It depends chiefly on the purpose of each concrete lesson and is determined by the nature of the study material examined during the lesson and by the degree of the pupils' familiarity with it. In our experiment independent work was used at various stages of the study of new material, but still the proportion of independent work was especially great during the consolidation of knowledge acquired by the children under the teacher's guidance[1] and was less during lessons devoted to the study of new knowledge.

For guidance let us note that in our experimental class the number of cases of independent work varied according to the concrete content and purpose of the lesson from one to five-six (when the main aim was to consolidate and review previous material).

Still, frequent switching of the children from one type of task to another, which was observed at various lessons of the Lipetsk teachers, may be to the detriment (if it is taken as a model for the everyday work with the pupils) of training the children's fixed attention and forming in them the skills necessary for lengthy independent work. However, the ratio between the children's independent and non-independent work in our experience generally appeared to be approximately 1:2 (32%) during the entire period of the experimental instruction (grades 1 and 2); and during individual lessons the children's independent work took 8-10 to 30-35 minutes.

We turn to the question of the concrete position of independent work for children in arithmetic lessons. If the minutes of lessons are examined from this standpoint, it can be observed that independent work in the public school is now being conducted, as a rule, after the explanation of new material, as a consolidation or as a check of the children's mastery of previous material, as well as for review. However, by examining the work of the best teachers we conclude that the pupils' independent work at arithmetic lessons can and should find its place in checking homework assignments (experience of teachers in Leningrad,

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[1] In evaluating the results of comparing lessons in Lipetsk schools with those of other regions, one must take into consideration that all 32 lessons observed by the co-workers of the Academy (A.S. Pchelko and myself) in the Lipetsk Region were based on material already familiar to the children. We could not observe any lesson in which new material was introduced. Consequently the pupils' independent work in the Lipetsk experiment was observed only at the stage of consolidation.

Moscow, Lipetsk), in conducting oral calculation, and in preparing to examine new study material. This experience has already been described in our literature.

We shall therefore concentrate on examining only one very important but heretofore neglected question--the question of giving independent work to children who are learning some new study material.

Teaching New Material by Giving Children Independent Work

At the lessons we observed, children were seldom given independent work at the stage of their familiarization with new study material (in 6 of the 120 cases in Table 1 and not once at the lessons observed in the Lipetsk Region schools since the latter were always built around study material familiar to the children). Observations and conversations with teachers confirm that these indices are not accidental. There is a rather widespread opinion among the teachers that the study of new material through pupils' independent work is possible only in the upper grades. Authors of modern methodological guidebooks on arithmetic instruction also proceed from this implicit assumption. In disclosing the methodology of familiarizing pupils with one or another new mathematical fact, they limit the exposition, as a rule, to a detailed description of what the teacher should explain and how he should explain it.

True, it is often a matter of which devices and means of work help raise the pupils' activity, attract their attention, awaken their interest in the problem being examined, etc. The skillful use of visual aids, demonstration of the practical significance of the problem being considered, careful selection of material on which the explanation of a new operation or concept is based, skillful construction of the discussion with the pupils, and other devices or means of work may be used in familiarizing the children with new material. However, it is almost never mentioned that it might not be impossible, if only in a few cases, to ask the children to try by themselves--completely independently--to gain an understanding of a problem which is new to them. Moreover, the final goal of elementary education is not only that the children learn the information imparted by the teacher, or even that they learn how to consciously apply it--it is very important that the pupils who have finished school be sufficiently prepared for the independent mastery of new knowledge.

It is perfectly clear that the ability to independently master new knowledge will not appear spontaneously. It must be trained little by little, gradually and systematically. Is it right to begin such training, let us say, only in the fifth grade? Is there sufficient justification for relinquishing an earlier introduction of the devices and methods of instruction which the children would need for independent solution even of very uncomplicated, easily understood, but still unfamiliar problems?

Let us consider the solution to the problem of the forms of tasks dealing with new material proposed in one of the latest manuals of methodology (published after the promulgation of the school law) [17]. In a special section, "An Explanation of the New Educational Material" Polyak wrote:

> Often a new operation presents itself as a complication of one mastered earlier. Thus, the addition and subtraction of many-digit numbers presents itself as a complication of the addition and subtraction of three digit numbers, multiplication by a three-digit number as a complication of multiplication by a two-digit number, etc. Upon the introduction of each new operation it is advisable that the pupils independently carry out that part of it with which they are already familiar and that only the new material be explained to them. Thus, when introducing multiplication by a three-digit number, it is possible, after an explanation of the decimal structure of the multiplier and a scheme for carrying out the operation, to ask the pupils to independently multiply the multiplicand by the units and tens of the multiplier; the addition of the three partial products can also be carried out independently by the children. Thus, only the new elements of this operation are explained here, significantly increasing the participation of the children in its mastery. But even the new elements should be explained so that the children appear to consider the problems along with the teacher, so that they themselves, under his guidance, seem to find the means of solution [17:24-25].

The recommendations quoted above are directed against that clearly harmful practice, which, unfortunately, to this day still occupies a place in the work of some teachers who consider it necessary to explain to the children everything from beginning to end--not only what is really new, but even what is already familiar to them. However, even here the author limits the independent work of the children to the reproduction of what has already been mastered. Even for a minimal step forward, which must be made, for example, at the transition from the addition of three-digit numbers (already known to the children) to the addition of many-digit numbers (with whose decimal structure they are also already familiar), the author considers it impossible to entrust to the pupils themselves. It is proposed that the teacher explain this new material and that the children be asked to follow his explanation attentively, since later on "they will be asked to give just such an explanation."

Is it not possible to organize the work in a different way, so that from the beginning the children attempt to gain an understanding of an unfamiliar instance, the understanding of which has already been fully prepared for by all the preceding work and so that the children do not appear to consider and do not "seem to find, by themselves, the means of solution", but really consider and search for it independently? We found experimentally that such tasks are within the capabilities of younger school children. As a result of our research, it was established that with appropriate methods of instruction, independent tasks for children can be used in the introduction of new arithmetical problems even in the first year of instruction. Necessary prerequisites for its employment are: (a) systematic work in accordance with the children's development on the practice of independent tasks--the systematic but very gradual increase of the demands made on the children to independently compare and generalize from observed facts; and (b) the ability to establish a connection between what is new and what has been mastered, to transfer knowledge and skills acquired earlier to the solution of a new task.

Let us examine several concrete examples indicating exactly how we fostered the pupils' capacity to generalize, and how they afterwards made use of this capacity when new arithmetical facts were introduced. It is common knowledge that first graders meet great difficulties on their first acquaintance with numbers. The formulation of abstract concepts about numbers is based on the examination of a great number of concrete facts--practical dealings with groups of objects. However, even here--literally from the first step of instruction--our goal was to focus the children's attention on the numbers themselves. Thus, by introducing his pupils to the formation of the number two as the addition of a unit to one, and then of the number three as the addition of a unit to two, of the number 4 as the addition of a unit to three, etc. (of course, at first using objects and only later on at an abstract level), the teacher tried to make the children aware that each number can be arrived at by adding a unit to the preceding one. This was accomplished in practice in the following manner. At every class devoted to the introduction of new numbers, before examination of the formation of a new number we went over (as a preliminary) with the children the means of arriving at numbers mastered in preceding lessons. In the lesson devoted to the number five, for example, the teacher put a circle on the demonstration apparatus and asked:

> How many circles did I put out? What must we do to get two of them?

When she received an answer the teacher put another circle on the board.

> How many circles are there now? (two circles)
>
> How many circles will there be, if I put out another one? (three circles)
>
> And what should I do to make four circles? (Add one more to the three circles.)

Then the children's attention was directed to the row of figures on the apparatus (1,2,3,4,). The teacher, pointing to the figures, asked the children the following questions:

> How much must we add to one in order to make 2?
>
> How much will there be if we add one to two? How much greater is four than three? How much must we add to three to make four?

After she had received answers to these questions (the questions did not present any particular difficulties to the children, since they had been reviewed at the study of each number) the teacher asked:

> What number comes after four? How much must we add to make five?

When she had received the correct answer to this question (formulated in abstract terms), the teacher shifted the work to the level of concrete operations with objects. She asked the children to count out four circles (from the didactic cut-out material which each pupil had) and to lay them in a row on their desks. When she had ascertained that everyone had executed his task correctly, she said:

> Now, make it so that there are five circles. What must be done for that?

Then analogous exercises were performed using other demonstration and individual material (counting sticks, the beads of the class abacus and others). The results of the work were summed up with the questions:

> What number comes after four? How much must we add to four to make five? How did we get the number 5?

We checked in a preliminary way this whole program of work by using individual experimental lessons with pupils in the class. Children were selected for the experiment who, according to the preliminary investigation (carried out before the beginning of school) were less well prepared than the others. In the experiment, we went ahead a little. During the period for individual tasks we asked the children such questions as:

> What number comes after five? How can we get this number? Count out six sticks. What must be done to get seven of them? etc.

With the preparation described above, even the most poorly prepared children were able to answer these questions. Thus, the appropriate generalization was within their power of understanding.

Let us consider another example. During the study of each new number a great deal of time is devoted to the consideration of its composition. Usually the number's composition, just like its formation, is considered in class in always the same fixed way. The lesson begins with practical exercises—the children lay out the assigned number of objects in two piles, name several variations of such a division, and then examine the drawings in their textbooks which illustrate all the possible combinations of the given number into two addends. Finally, they repeat all this on an abstract level.

As we were observing the progress of these lessons in the classes of various teachers, we became convinced that children (with rare exceptions) are not capable of independently noting a consistency in the enumeration of all the possible combinations of addends of a number, which would make it possible not to lose sight of a single one of them. Usually they name the various combinations in no definite order. Thus, some combinations do not get named and others are repeated. As the children proceed to the study of each successive number up to ten, no essential change is evident. Moreoever, since the number of various combinations become larger and larger for each successive number studied, the reproduction of all of them from memory (which many teachers require) become more and more difficult for the children.

It is perfectly clear that this task would be significantly facilitated if it were possible to teach the children a system for the enumeration of all these instances. In relation to this the following questions occurred to us: "Would it be possible to do this? Would such a requirement be too difficult for first graders? Would it require too much time?" At first, during the individual experiments and later during the trials with the class, we tried out various forms of the task as well as methodological devices directed toward imparting the proper capacity to the children. As a result, we selected a particular system of exercises organically included in the lessons and devoted to the mastery of the composition of the numbers studied. This system of exercises, used in the experimental classes of school number 315 in Moscow, made it possible to increase, from lesson to lesson, the amount of the children's independent participation in examining the composition of the new numbers, and to gradually accelerate the shift from practical operations with objects to an examination of the composition of the number on an abstract level. This task proved to be good preparation for the study of addition and subtraction.

Thus, the composition of the numbers 2,3, and 4 was discussed on the basis of practical operations with concrete objects. The children were asked to take in both hands the proper number of sticks and to lay in two piles the proper number of circles, squares, etc. Then they were shown with the demonstration materials how to make gradually and successively, all the possible combinations of addends. The teacher for example, put out four circles on the upper shelf of the demonstration apparatus and then removed one circle at a time from the upper shelf and put in on the lower shelf. Each time the children named the proper groups (four--three and one; two and two; one and three) Each time the proper figure was placed on the shelves of the demonstration apparatus. Thus, as a result of the examination of the composition of the number four, the following array of numbers was placed on the demonstration apparatus:

$$\begin{array}{cccc} & 3 & 2 & 1 \\ 4 & & & \\ & 1 & 2 & 3 \end{array}$$

Analogous tasks were carried out by the children with individual material. They made suitable sketches in their notebooks (outlining the appropriate number of squares and filling them in with pencils of two colors). In conclusion, the teacher asked the children either to enumerate aloud all the combinations of addends whose sum equaled the number being studied, or to indicate them with the help of cut-out (movable) figures, taking care that these combinations were enumerated successively.

The study of the composition of the numbers five and six was accompanied by essentially the same tasks, but everything was done by the children themselves--the teacher showed them nothing, but called one pupil after another to the board and asked them to demonstrate independently what groupings could be made from 5 apples, 5 squares, and others. The children had to follow the answers of their classmates and correct them if they made any mistakes.

After examining the composition of a number with demonstration materials, the children were asked to indicate its composition using cut-out figures. Each pupil had to do this independently, although the teacher walked along the aisles and helped the pupils who needed it. At home the children independently drew illustrations of the composition of the numbers being studied.

The study of the composition of the numbers seven and eight did not even begin with a demonstration, but with the independent work of the children with individual materials. However, each step was examined collectively under the teacher's guidance. During the study of the number eight the teacher asked the children to carry out all work independently--to take eight circles and lay them out in two groups to indicate all the possible combinations of addends, labeling them with the aid of the cut-out figures. The result should have been:

$$\begin{array}{cccccccc} & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 8 & & & & & & & \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

This is a difficult assignment which demands great concentration and prolonged careful attention. Even those children who had grasped the principle of its execution had difficulty carrying it to a conclusion. The teacher had to give individual help to several pupils. When they had finished the independent work, these pupils were called to the board and had to indicate with the help of the demonstration material all the combinations of addends whose sum equaled the number eight. Meanwhile the remaining children checked their work.

The results of this preparation were evident at the next lesson. The repetition of the composition of the numbers seven and eight with which this lesson began presented none of the difficulties which had been observed in the other lessons. During the lesson on the composition of the number nine, the teacher conducted the following discussion with the children:

> On the upper shelf of the demonstration apparatus there are nine circles. How can I find out what pairs of groupings I can divide them into? What must I do? (Remove one circle and put it on the lower shelf).
>
> How many circles are there left on the upper shelf? How many are on the lower? (On the upper shelf, there are 8 circles and on the lower there is one circle.)
>
> And what shall we do now? (Move another circle from the upper to the lower shelf.)
>
> How many are there now left on the upper shelf? (7 circles.)
>
> How many are there on the lower? (On the lower shelf there are 2 circles.)
>
> Represent this situation with the cut-out figures, and then show the next division with the help of the figures.

After this lesson we questioned several children and were convinced that even the slowest in the class understood the main point--namely that all the combinations of two addends composing a number can be named without missing any, if one group is successively increased by one at the expense of the other. The only difficulty they encountered was in determining the size of the new groups. With the use of visual aids even these children were able to demonstrate and name all the combinations of two addends composing a number.

In the lesson devoted to the composition of the number 10, the teacher immediately asked the children to try to enumerate without recourse to visual aids the possible combinations of addends composing it. In this lesson, visual aids were used only to check the pupils' answers and for consolidation.

It is perfectly clear that the study of the composition of the numbers from 1 to 10 was not limited to the exercises described above--they constituted only a part of it. We dwelt at length on the description of them only in order to indicate how gradually children are brought to an independent mastery of new material through questions analogous to those relating to smaller numbers.

We strove to increase the children's independent participation in the study of new problems also in work with other educational problems. Thus, in the study of the topic "Addition and Subtraction of Single Digit Numbers" the children were made aware of the general principle which forms the basis for all these cases, namely that any number may be increased or decreased by 1 or by any of the groups of numbers composing it.

In the study of addition and subtraction of numbers totalling less than 20, both with and without carrying, the first few examples were based on a detailed explanation by the teacher, but the independent participation of the children was elicited more and more in the course of the demonstration and explanations. In the course of this work, the teacher tried to get the children to grasp the general principle on which each concrete solution was based. With such preparation it proved possible to carry on the consideration of new instances, analogous to those learned earlier, based on the children's independent work.

Still greater possibilities for reliance on children's independent work when introducing new material were discovered in the next grades, as the range of numbers studied is gradually broadened. Having thoroughly mastered the basic methods of calculation with numbers under 20, the children have acquired almost all the knowledge necessary for carrying out the addition and subtraction of numbers totalling less than 100. Therefore, in second grade we almost always start from the children's independent work, in the study of new cases of addition and subtraction. We need only direct this work, choosing appropriate visual aids, controlling the process of its execution, giving it the proper direction, and helping the children summarize the work (i.e., clearly formulate the rule that they had used when solving one or another example). In the third grade we conducted the study of the topic "Million" in one of the experimental classes (teacher T. V. Titova) so that the children's independent work served as the starting point for the explanation in almost every lesson devoted to addition, subtraction, multiplication and division of numbers less than 1,000,000. The pupils had to try independently to gain an understanding of the application of a familiar computational operation to a broader range of numbers. In another class (teacher M. A. Korosteleva) the introduction of the new material was conducted as recommended in G. B. Polyak's manual quoted above.

In this way the following picture of the study of new material was developed. In the class where the teacher explained everything to the children, the work progressed very calmly, the children almost never met with any difficulties in the understanding of the new material, but it did not arouse any great interest in them. The children met with difficulties only when they had to apply independently the rules explained by the teacher to the solution of new examples and to the explanation of the derived operations. We came upon such phenomena more than once in this class later on in the study of subtraction, multiplication, and division. In the other class, difficulties arose immediately--as soon as the children were asked to try to understand independently a new example and to prepare an explanation of its solution. The progress of this work is illustrated using, as an example, the explanation of the addition of many digit numbers.

The teacher began with an example, familiar to the children, of the addition of three digit numbers totalling less than 1000. The solution of this example was put on the board with a full explanation by one of the class's average pupils. Next to the solution on the board was hanging a chart of the decimal places. Several exercises were given in the writing and reading of many-digit numbers using this chart. Afterward the teacher said:

> If all of you have learned how to explain the solution of such examples and remember the names of the decimal places, you will be able by yourselves not only to solve a new example on the addition of many digit numbers, but also to explain it. Now I am going to give you such an example. Solve it without rushing, explaining each step to yourselves: the significance of the numbers you are adding, what the results are, what you are keeping in mind, what you are writing down. When you are ready to explain the whole solution raise your hands.

On the board, in the columns of the numeration chart the teacher wrote the example 2347 + 6485 and the children started their independent work. After two minutes, five pupils raised their hands; after another three minutes--18. We walked along the aisles--only four had not written down the solution to the problem. Two of them had simply not had time to write it down, because they hadn't started working at the same time as everyone else; but the two others asked in bewilderment "How can I solve it?" We had to talk with them separately. Five minutes after the work had begun, the teacher asked, "Who has not yet solved the example?" It turned out that everyone had solved it. After this, the teacher called on four pupils one after another ("in a chain") to give the necessary explanation of the solution. During the explanation they were permitted to look at the chart of decimal places and at the example written on the board. All the explanations were correct. Nor did the next example (which was solved with an explanation at the board) present any difficulty.

We attribute this success to the fact that (due to the teacher's instructions) the children understood from the very beginning that in their explanations of the example's solution they were to be guided by the knowledge they had gained from the study of addition totalling less than 1000 and from their knowledge of numeration. It is also important that the requirement of gaining an understanding of the new material independently "without the teacher's help" always arouses heightened interest in children. The fact that the example was written in the columns of the table of decimal places also proved of great help to them.

An examination of the children's mastery of the new material in succeeding lessons indicated that the children gained greater mastery of the new material when they tried to understand it independently from the very beginning, and received help from the teacher only if they could not find a solution no matter how hard they tried. Thus, the possibility of children's independent work on the introduction of new educational material is dependent on the fact that during the shift from one cycle to another, the pupils work with material which is similiar in many respects, and that the essence and methods of execution of arithmetical operations remain the same, no matter what numbers are involved. Moreover, a greater and more daring reliance on the active independent working of the pupils' minds is also made possible by the fact that during the learning process children gradually amass knowledge of a great number of varied facts, paving the way for and sometimes spontaneously leading to one or another generalization, or to the deduction of new laws.

One of the greatest advantages of children's independent work appears when, guided by the knowledge of numerous facts, they independently make new deductions. One must approach the construction of such tasks carefully, estimating their difficulty, and preparing the material and assignments thoroughly.

Consider the following example. Starting in the first grade, when children, in the study of the addition of numbers less than 10, are introduced to the commutative property of addition, they constantly use this property in oral calculations. The recommended method (which has been widely put into practice) is the rearrangement of addends during the study of each new instance of addition, where this seems advisable. However, the formulation of the commutative property, and the introduction of its application in the verification of solutions to examples in addition, are not given until the fourth grade.

Let us consider the explanation of the teacher E. G. Khvatova in the lesson described in her article "Devices Which Increase the Efficacy of an Arithmetic Lesson" [6]. After a check of the homework and oral recitation, during which there was not one problem requiring the rearrangement of the addends, the teacher turned to an explanation of new material.

> I begin the explanation by writing the first addend as 14, and the second as 18. I ask for the sum (We write down 32). I give the second example, but this time with the addends rearranged. The children solve it. Four more examples are solved: 27+55= 82; 270+320=590 and 55+27=82; and 320+270=590.

I ask the children to look carefully at each pair of examples and to say what they have noticed. They all look, think, raise their hands and give their conclusions.

The first addend becomes the second and the second the first, but the sum is the same.

The addends changed places, but the sum stayed the same.

I confirm the correctness of the conclusion. The pupils make up their own examples and repeat the conclusion. Then I ask them to read the conclusion on page 53 of the textbook.

Let us check this rule with large numbers.

Then they solve several examples in the addition of many-digit numbers with checks. In other lessons, just as in this one, I strive to activate the children's thought processes and their independence in work.

What can we say about this lesson? The teacher clearly strives to organize the work around the new material so that the children are as active as possible, so that they independently observe, think, draw conclusions, etc. And this is correct. This aspect of the material of the lesson is very gratifying, since preceding instruction laid a firm basis for the children's conscious perception. However, the teacher did not succeed in sufficiently realizing these possibilities. The reason is that a connection between the new material and what the children already knew was not established in the lesson. This teacher's lesson conforms to the classical scheme: from the children's independent observation of a number of facts, and the analysis and comparison of them, to an independent conclusion, perfected in subtlety and accuracy of formulation under the teacher's guidance, and then consolidated with new examples composed by the pupils themselves.

In many cases one could not object to this method, because, as a rule, it demands the strenuous and independent working of the children's minds and furthers their development and understanding and mastery of the new material.

The task, which is rather complicated and demands a clear formulation, was set up rather well by the teacher. The number of examples chosen was sufficient for the formation of a conclusion, and they were appropriate to the task; the questions were clearly formulated; the children were given time for reflections; etc. The presentation would be acceptable, if all this work did not ignore the fact that the pupils had been familiar with the commutative property of addition since the first grade. It is self-evident that, under these conditions, any consideration of this problem in the third grade should be carried out with extensive use of what is already known, and that the children's knowledge should serve as a starting point for further progress.

To prepare for an explanation of the way to check examples in addition (in essence, this is the only new material in the lesson) there should be a preliminary solution of several examples in which the use of the commutative property of addition facilitates computation. This task can be set up both directly (in the form of oral recitation) and as independent work for the children. It is important only that before beginning work on the new material the children be told once more that, in the addition of two or more numbers, it is not obligatory to perform the operation in the order in which it is written and that the addends can change places without changing the results.

After bringing this to their attention for the cases involving small numbers, to which the property had formerly been applied, it is possible to ask the children to check independently its applicability to larger numbers. After this rule has thus been formulated and extended, the teacher can put the following question to the children

When and why do we use the commutation of addends?

The children will answer:

When the rearrangement of addends *facilitates computation*. In the next step the teacher tells the children that this device may be used not only to facilitate computation, but also *for checking it*, and asks them to think about how the rearrangement of addends can be used for this purpose. The children will be sufficiently prepared to answer this question. The teacher's job is to help them formulate the appropriate rule and devise exercises to consolidate this knowledge, as in the lesson described above.

By using this example (taken from the work experience of a teacher) our goal was to indicate that children's independent work with new material, when prepared for by previous instruction, can be successfully conducted if, in the lesson devoted to the study of the new material itself, a connection is established between the new material and what the children already know. All the children's knowledge, capabilities and skills in the area should be initially brought into play. As for the required content and character of the independent tasks given to the children, the most important thing is that they direct the mental activity of the child to the solution of a new problem which is within his capabilities, the solution of which is based on facts which are already known to him from past experience, and that furthermore, the independent tasks require him to make observations, comparisons, and analyses within his capabilities. The best teachers successfully carry on this sort of work in the third and fourth grades. Take, for example, an exerpt from the class record of a lesson given by the teacher A. V. Kozokina (School No. 315 in Moscow). The lesson concerned the rule for the checking of multiplication by division. The teacher started work on the new material by asking the children to try to understand independently, assignment number 164 in the textbook. Here is the assignment.

> Solve the following examples. What is the result when a product is divided by one of the multipliers?

$16 \times 5 = 80$
$80 \div 5 = 16$
$80 \div 16 = 5$

$x \cdot 6 = 120$
$120 \div 6 = 20$
$x = 20$

$8 \cdot x = 240$
$240 \div 30 = 8$
$x = 30$

Before the children began to work, the teacher put the following questions to them.

> Look at the first line--what do we call the number 16? 5? 80? Read the second line--what was the number 80 in the first line? (Product.)
>
> And the number five? (Multiplier.)
>
> Now by yourselves, look carefully at the results of this example and the next and then consider the columns and be ready to answer the questions I asked.

When many hands had been raised in the class, the teacher asked who had not yet understood what had to be done and could not answer. There turned out to be two such pupils. A discussion was conducted with them in which the other children also participated, consisting of the following questions.

> Read the first line naming what each number represents in this line. (16-multiplicand, 5-multiplier, 16 multiplied by 5 makes 80, 80 is the product.)
>
> Right. Read the second line. (80 divided by 5.)
>
> That's enough. What is the number 80 in the first line? (Product.)
>
> The number 5? (Multiplier.)
>
> What happened when you divided the product by the multiplier? (The result was 16, the multiplicand.)
>
> Tell me what is going on in the third line and what the answer is. (The pupil's answer follows here.)

Question to the class:

> What is the result when we divide a product by one of the multipliers?

Two pupils, one after another gave a complete answer to the question.

Class assignment:

> Once again look carefully at the next columns and tell me when we use this device. (When we solve examples where one of the numbers is missing.)

> Who can put it better, more exactly--what examples, what number?

The pupil who was called on gave the correct formulation.

> How can this device be used for checking multiplication?

She called on the slowest pupil in the class for an answer; although the answer he gave was not completely precise; it displayed correct understanding.

> You must divide by one of the multipliers and you will get the other.

The teacher gave a more accurate formulation and then asked the children to independently do exercise number 165, which required the solution of a number of multiplication examples and the checking of these answers by division. Five minutes after the start of the independent work, two pupils were called to the board to write the solutions to the first examples and their explanations. After a detailed analysis of these two examples and a review of the formulation of the rule, the rule was read from the textbook.

We see that in this lesson the study of new material is based on the children's independent work, and the pupils as a whole manage it very well. The explanation of the material to the two pupils who found the task difficult took place after all the others had successfully completed it independently and could take part in the explanation. Further work was based on the independent application of the derived rule to new instances and the analysis of them was given at the board only after all the pupils had done the necessary work themselves and the teacher was satisfied that everyone had managed it. The only purpose of the analysis at the board was a further repetition of the connecting explanation to consolidate the necessary formulation in the minds of the pupils. The analysis was conducted by the pupils--the teacher only guided them, and, where necessary, called on other pupils to eliminate a few rough spots from the formulation.

Children's independent work can serve as the starting point and basis for the consideration of new educational material not only when what is involved is the transference of a method of operation learned previously to a wider range of numbers, and not only when the way has been paved for independent generalizations by previous experience. In some cases the work can be so organized even in the study of a new kind of problem. We will take as an example the execution of such work in a lesson (experimental second grade of School No. 315, teacher L. E. Zaikina) devoted to the children's introduction to a new kind of problem, the addition of several units to a number in two operations. This is the problem.

> In the first there are so many, in the second there are so many more than in the first, and in the third there are so many more than in the second. How many are there in the third? (Problems 187-190 in the second-grade textbook).

In the lessons when observed at various times and with various teachers, the explanation of the solution of the problem was carried out in accordance with the description of this work in the manual by N. V. Arkhangel'skaya and M. S. Nakhimova [1:20-21]. We cite this description in full so that it will be clear what changes we effected in the methodology of the study of new material in our experiment. In the course of the description the questions which occurred to us while we were observing and analyzing these lessons are noted.

1. The teacher calls on two pupils, who stand in front of the class. Calling on a third pupil, the teacher tells him:

> Give Kolya five pencils and Fedya three more than that. How many pencils does Fedya get?[2]

2. Then the teacher calls on three pupils and tells a fourth:

> Give Volodya three pencils, give Yura two more than Volodya and Lev four more than Yura.
>
> So you'll be able to remember how the pencils were given out, I'll write it on the board. She writes: V: three pencils; Yu: two more pencils; L: four more than Yura.
>
> How many pencils did Yura get? How many did Lev get?

---

[2] Is it really necessary to dramatize this simple problem and to make such great use of visual aids during its solution, if we take into account that the children have been familiar with the addition of several units to a number since first grade and have already solved many similar problems without direct recourse to visual aids? M.M.

The pupils repeat the problem.

After distributing pencils and receiving an answer, the teacher dwells at length on the question of how the pupils figured out how many pencils each boy received.

Is it possible to find out right away how many pencils Lev received?

It becomes clear that first it was necessary to give the pencils to Yura and calculate how many he had received and only then was it possible to give the pencils to Lev and calculate how many he received.

The teacher writes the solution on the board.[3]

In the next stage of the lesson the teacher carries out the third task in exactly the same way:

> Give three pupils notebooks--to the first two notebooks, to the second three more than to the first and to the third two notebooks more than to the second.

Finally the fourth task is carried out.

> On the class blackboard circles should be drawn in three lines--on the top line six circles, on the middle line three more than on the top, and on the bottom four more than on the middle. How many circles should be drawn on the middle line and how many on the bottom line?

Further along in the plan a description of the whole course of this work is given from which it is evident that both the drawing and the writing of the solution on the blackboard are done by the teacher himself. In practice the teacher more often calls on pupils to do the drawing and the writing of the solution.

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[3] Observations of the work of the teacher and children at the lessons indicate that the first part of the task--the practical solution of the problem--does not cause the children difficulties. Difficulties are encountered only when the teacher turns their thoughts to the beginning of the task and requies an analysis of each step of the solution, when he asks for a translation of the problem's solution into the language of arithmetical operations. With these methods, this part of the task takes up much of the pupils' time and effort and in the end the teacher must write the solution of the problem himself. In connection with this one asks whether it is necessary to separate in such a way the children's practical solution of the problem from the writing of the solution. Would it not be more appropriate to fixate each step of the solution, so that the children would not have to return to an analysis of their actions after the problem is already solved?--M.M.

After this the curriculum dictates that the children carry out two assignments given by the teacher.

1. The teacher asks the pupils to put sticks in three rows--in the top row, seven sticks, in the middle row four more than in the top, and in the bottom row, three more than in the middle. The assignment is to find out how many sticks there are in each row and to state how this can be determined.

2. The teacher has the pupils draw crosses in their notebooks--on the left, four crosses; in the middle, two more than on the left; and on the right, two more than in the middle. The assignment is to write the correct number under each group of crosses.

Only now does the teacher turn to the analysis of problem number 93 from the textbook, which is completely and concretely illustrated.[4] This is how the analysis of the problem looks:

> The pupils read the problem and examine the illustration. Then the teacher puts the problem on the board in an abbreviated form:
>
> Bottom -- 10 books
> Middle -- 6 books more than on bottom
> Top -- 4 books more than on middle
>
> An analysis of the problem is conducted.
>
> What is asked in the problem?
>
> Is it possible to find out, at once, how many books are on the top shelf?
>
> Why is it impossible? What is it possible to find out from the first? And so forth.

Then comes an oral analysis of the solution to still another of this sort of problem (the children write the solution at home). Thus, both the authors of the curriculum and the teachers who follow the recommendations cited above consider it necessary that the children analyze two problems of a new type also with the aid of dramatization and finally, that they solve yet another problem under the teacher's guidance, with the full illustration of the whole course of its solution put on the board by the teacher. Only after all this has been done do they consider it possible to turn to the children's independent work with individual visual aids. But even at this stage, each step in the children's work is controlled by the teacher and is explained to the children with his help. It is also proposed that the assignment which the children do in their notebooks be executed with the full use of concrete visual aids. The next problem (from the textbook) is still not to be given to the children for independent solution (moreover, its conditions are illustrated in the book) but is to be analyzed collectively from beginning to end, and the children must only write down the solution.

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[4] In the book quoted, this problem is listed as number 187 (See [14]).

However, as has been noted above, before the introduction of these problems, the children have already solved many problems which required them to find a number bigger than another one by several units; they have also solved compound problems. Thus neither the fact that in order to solve the problem it is first necessary to find some missing information, nor the necessity of increasing a number by several units is new to the children. By this time the pupils should be able to solve problems on the addition of several units to a number without recourse to visual aids.

All these considerations, supported by the data gathered in our observations of pupils' class work, gave us grounds for assuming that the share of the children's independent participation could be substantially increased. As always, we first checked this supposition during separate lessons with individual pupils. The sessions continued up to the introduction in class of the problems under consideration. The aim of introducing the problems was to ascertain the difficulties children would encounter in the solutions, and, in accordance with the difficulties, to outline a method of instruction which would minimize these difficulties and provide a faster transition to the independent solution of the problem. Proceeding from the fact that the successful solution of this new type of problem requires a firm knowledge of simple problems in the addition of several units to a number, we selected for the sessions only students who, up to this time, had been able to handle independent work in the solution of such problems in class.

This work consisted of three tasks.

> 1. Outline six squares on one line, and two more than that on the next line.
>
> 2. Solve the following problem.
>
> In one piece there were 30m. of material and in in another 10 m. more than in the first. How many meters of material were there in the second piece?
>
> 3. Formulate the question and the solution for the following problem.
>
> Kolya solved 5 examples and Tanya solved 2 examples more than that. Then write down a complete answer to this question.

Such tasks were carried out more than once in drills conducted in the first grade. During the session we also began with a practical assignment. Draw four circles on one line and two circles more than that on the second. Each time we asked the child why he had drawn six (or nine) circles. All of the six children with whom we had sessions were able to handle this task. Then we assigned the following problem from the textbook.

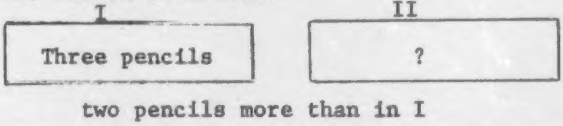
> In one box there are 3 pencils, in the second there are two more than in the first, and in the third there are four pencils more than in the second. How many pencils are there in the third box?

We assigned such a problem for independent solution in the first sessions conducted with good students in order to find out what difficulties the children might encounter. We found out from this that the errors committed during the solution were connected not with choosing an operation, but with insufficiently analyzing the conditions of the problem. Thus, one pupil ended his solution after having carried out only the first operation, and another, in both operations, added first two and then four pencils to three pencils.

Because of this finding in the sessions with the next pupils (for these lessons slower pupils had been chosen) special attention was devoted to the analysis of the conditions. First, we reminded the children of the form for the schematic representation of the conditions which was used in the solution of simple problems on the addition of several units to a number, and then asked them to represent the following problem diagrammatically.

> In one box there are 3 pencils and in the second there are 2 pencils more than that. How many pencils are in the second box?

After they had written down:



we asked the children to write down the solution to the problem.

A compound problem then was assigned:

> In one box there are 12 candies, in the second there are three more candies than in the first, and in the third there are six more candies than in the second. How many candies are there in the third box?

The problem is read independently. Then the following questions were asked:

> How many boxes are there in all? (Three.)

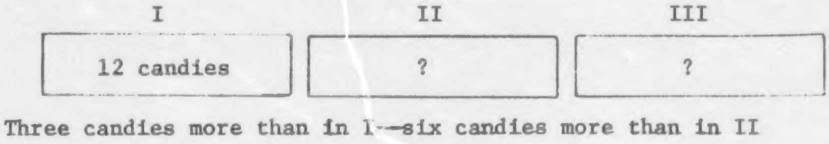
> Make three rectangles and write in them as much as you know from the problem about the number of candies in each one.

If this task caused the pupil any difficulty, we helped him by asking leading questions.

> Do you know how many candies there are in the first box? Write it down in the first rectangle. Does it say in the problem how many candies there are in the second box? (No.)

> Put a question mark in the second rectangle. Does the problem say how many candies there are in the third box? Now write down what is said about the second and third boxes.

As a result the following representation was written down:



After this was written down, we asked them to solve the problems independently and again observed how the children handled the work and what caused them difficulties.

We were prepared to help the children in the final analysis of the problem. But, in practice, this was not necessary. All four children with whom we conducted these individual lessons solved the problem by themselves. It is true that when we asked them to explain why it was impossible to find out at once how many candies there were in the third box, not everyone could answer precisely enough, but it was evident that the solution had been carried out with an understanding of the essence of the matter. Here are the answers.

> Slava K: First I figured out how many were in the second one, and then how many in the third.
>
> Misa M: There are two problems here. First 12+3 makes 15 candies, and 15 plus 2 more make 17 candies.
>
> I asked: But why did you add the 2 candies to the 15 candies?
>
> Misa M: Fifteen--that's in the second and in the third there are 2 more.
>
> Tanja M: Because in the third there are 2 candies more than in the second, but how many are in the second must be figured out first.

Then problem number 187 from the textbook was considered. The children had to read it through themselves, examine the drawing in the book, and write down the solution. And again everyone was able to handle the problem. Only Yura Sh wrote down both operations in one line:

10 books + 6 books + 4 books = 20 books

Individual lessons with six pupils, of whom four were among the slower pupils of the class, convinced us that problems of this type are not so difficult that their explanation requires the prolonged work that N. V. Arkhangel'skaya and M. S. Nakhimova recommend. In accordance with our formulation, the teacher L. E. Zaikina conducted this lesson in an experimental class in the following manner.

At the beginning of the lesson, in the checking of the homework and oral calculations, two problems were solved orally—one on increasing and the other on decreasing a given number by several units. Then the pupils were asked to compose a problem to fit a diagram and then to solve it. For this composition two variants were given:

I		II	
$\boxed{10}$	$\boxed{?}$	$\boxed{\text{Six}}$	$\boxed{?}$
Three more		Four more	

We cite examples of the problems the children composed:

> In one box there are 10 kilograms of apples, in another there are 3 more than that. How many kilograms of apples are there in the second box?
>
> Brother found 10 mushrooms and sister found three more than that. How many mushrooms did sister find?
>
> In the garage there were six cars and in the parking lot there were four cars more than that. How many cars were in the parking lot?

Each pupil wrote down in his notebook the solution to the problem he had made up. The teacher walked up and down the aisles and checked the correctness of the assignment's execution ; then she called on three pupils one after another to read their problems and explain the solutions. The rest of the children listened and checked the answers. For the solution of one of the problems another pupil was called on. Then the teacher wrote on the blackboard an assignment to be worked independently in the notebooks.

> Outline squares:
> On the first line--four squares
> On the second line--two squares more than on the first

After this assignment was carried out, the teacher asked:

> How many squares are outlined on the first line, how many on the second, why six?

When she had gotten answers to these questions, she added to the assignment by writing on the board:

> On the third--three squares more than on the second line.

This additional assignment was read by one of the pupils. The teacher once more emphasized the condition with the question:

> Three squares more than on which line?

The children answered that they were to outline 3 squares more than on the second line. After this assignment was carried out too, the teacher again asked how many squares there were in each line and why.

After this the children were asked to listen attentively to a problem and to compose independently a diagram for it.

> In one box there are four pencils and in a second there are two pencils more than that. How many pencils are there in the second box?

The problem was repeated by two pupils, and then the children turned to its diagrammatic representation.

During the execution of the assignment the teacher had to help two pupils. When almost all the children had handled the task, one of the pupils was called on to draw the appropriate diagram on the board. The rest of the children checked to see if they had done the diagram in the same way. The class was given the question:

> Do we know how many pencils there are in the second box? (No.)
>
> Can we find out? (Yes.)

Then the teacher continued the problem--"And in the third box there are three pencils more than in the second. How many pencils are there in the third box?"--and called on one of the pupils to complete the diagram on the board. The class was given supplementary questions about the diagram.

> Why did Jura put a question mark in the third rectangle? (Because we do not know how many pencils there are in the third box.)
>
> What does the problem say about the third box? (It says that in the third box there are three pencils more than in the second.)

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The words "in the second" were underlined on the board. The children completed the diagram in their notebooks.

Then the following question was asked of the class:

> Who will be able to figure out how many pencils there are in the third box?

Many hands were raised.

> Who does not know how to solve the problem? (Two hands are raised.)
>
> Solve it, children.

The teacher goes over to the children who had hesitated. One of the pupils who had solved the problem incorrectly was subsequently called to the board and under the teacher's guidance and with the help of his comrades carries out a complete analysis of the solution.

The next stage of the work was the solution of problem number 187 from the textbook.[5] The children independently read the conditions and examined the drawing for the problem. The teacher asked them to compose a diagram independently. Before the children began the task she asked them how many rectangles would be in their diagram and showed the most convenient way to draw the rectangles on the board (top, middle, bottom). After the assignment was carried out, one of the pupils drew the diagram on the board.

The course of solution is outlined in the direct questions asked the class:

> Can we find out at once how many books are on the top shelf? Why? What must be found out first?

One of the pupils repeats the plan of the solution:

> First we figure out how many books there are on the middle shelf and then how many books on the top shelf.

The solution is written down by the children independently and checked collectively. Then problem number 188 from the textbook is assigned as an independent task.

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[5] Here and later we refer to the standard textbooks: [13, 14, 15, and 16].

> The first strip is 5 cm long. The second strip is 3 cm longer than the first. The third strip is 2 cm longer than the second. Compute the lengths of the second and third strips.

All the children were able to handle this task. Thus we see that by making considerably greater demands on the children, for which, however, they were properly prepared, we succeeded in constructing all the work of the lesson on the foundation of the pupil's active independent work. The lesson turned out to be more interesting and livelier for the children, and the children consciously mastered the material.

The pupils' independent work served as the starting point for the examination of several other types of problems in the second grade—the explanation of problems on "the reduction into units" and its inverse and others.

Pupils' independent work can sometimes be used even when first introducing children to instances of arithmetic operations that are new to them in principle, if the teacher succeeds in selecting the exercise material or constructing the visual aids to lead to the right means of solution. We demonstrate this by a practical example from teaching in our experimental class. Subtraction of one-digit numbers within the limits of 20 without carrying over ten was introduced to the children.

Beginning with the consideration of new cases of subtraction of the type: 16 - 2, 17 - 4, and so forth, the teacher, instead of turning immediately to an explanation of the new subtraction device (with a demonstration of the appropriate operations with demonstration material) asked the children to solve themselves the example 17 - 4 using counting sticks. Ten of each pupil's counting sticks had earlier been tied into a bundle. Thus, when they were asked to take out 17 sticks, each one had a bundle and seven separate sticks. Of course, when they were taking away four, none of the children tried to untie a bundle but used for this purpose the seven separate sticks. To the teacher's question; "How many sticks were left?"--the children confidently answered; "13 sticks were left."

Then came the questions:

> How did you find out that 13 sticks were left? How many separate sticks were there? Why were only 3 left?

In answering, the pupils, in fact, reproduced the chain of reasoning to which the teacher had wanted to lead them.

Still another example (15 - 3) was solved in the same way. Again, in the check of the independent work, the teacher posed the questions:

> How many were left? How did you find it out? How did you take away, from which sticks?

After this the teacher asked the children to figure out mentally how many are left if four are taken away from 19. A forest of hands went up, and the pupil called on not only gave the correct answer but also explained how he had computed it.

Only later, during the summary of the work which had been done, did the teacher (at the dictation of the children who were called on) again demonstrate the solution of an example of this type. In the course of the demonstration the formulation, which had been arrived at through the children's explanation of the solution, was specified, and a detailed representation of the solution was put on the board. The fact that the children, from the very beginning, had, during the execution of the assignment, followed the course which the teacher required, can be explained in this case by the selection of materials which in themselves led up to just this course. If the same counting sticks had not been tied up in bundles earlier, the method of addition and subtraction with which the teacher wanted to acquaint the children would hardly have succeeded in coming to their awareness on the basis of independent work.

This approach was also used to examine new methods of subtraction and in the introduction of addition and subtraction with carrying over ten. For this latter purpose we used the widespread device "The Second Ten." This is a demonstration device consisting of two rows of 10 pockets each, in which cut-out geometrical canvas figures can be placed. The device had already been used for the organization of pupils' independent work in the examination of numbering within the bounds of 20, in addition and subtraction in cases where one of the addends is equal to 10, and then in addition and subtraction without carrying over ten. With such preparation using this device (each child in the class had his own), the study of carrying over ten addition and subtraction within the bounds of 20 proved to be completely within the children's abilities. In all the cases considered, the lessons proved to be both more interesting and more productive than the lessons completely constructed around the teacher's explanations of new material, where the children only had to listen attentively, remember, and repeat his actions and words.

Described above are separate instances in which we succeeded in constructing the introduction of new material on the foundation of pupils' independent work. Through analysis of the numerous facts of this sort, which occupy a place in teaching experience in our experimental classes, we came to the following fundamental conclusion about the advisability of such an approach to the study of new educational material in arithmetic lessons in the elementary school.

The use of children's independent work as a starting point for the introduction of new educational material proves productive:

1. If the material which is to be studied in the lesson does not contain anything new in principle relative to what is already known to the children, if the solution of the new problem is a reinterpretation of what was mastered earlier or is an application, with some modification of earlier acquired knowledge, skills, and habits.

2. If the lesson contains a problem in generalizing several relatively simple facts well known to the children, with which they have dealt more than once in the past.

3. If the method of operation, the approach to the solution of the problem with which the teacher must acquaint the children in the lesson, can be presented to them by creating appropriate conditions in the lesson itself (by using appropriately constructed and selected visual aids, by organizing appropriate preparatory exercises and so forth).

4. If the examination of the new material can be constructed on the foundation of the experiences with practical operations with objects, which the children have acquired earlier in the school and out of it.

5. If the children have been sufficiently prepared for independent work with a book, and the new material is set forth in a form which they can understand.

The examples, cited above, of the study of the commutative property of sums in the third grade, the study of the addition of many-digit numbers totalling less than a million (after the writing devices for the addition of numbers totalling less than a thousand have been studied), belong to the first case. In the teaching of arithmetic in the elementary grades, there proves to be a significant number of such instances, especially since many of the questions in the elementary arithmetic course are considered concentrically. This is related to the study of arithmetical operations (which are studied first within the bounds of 10, then within the bounds of 20, 100, 1,000, 1,000,000 and finally with numbers of any magnitude), and to the formulation of several arithmetical concepts (for example, the concept of the difference of numbers, fully prepared for by the solution of problems on increasing and decreasing numbers by several units) and to the solution of arithmetical problems (for instance, in the beginning of the second year of instruction the children become acquainted with problems in two operations, one of which is either addition or subtraction, and the second multiplication or division, and before that, in the first grade, they have already mastered the skill of solving simple problems with all operations, and problems in two operations with addition and subtraction). As an illustration of the second case, we may consider the introduction of examples in subtraction with zero in the answer and of the commutative property of sums to pupils of the first grade; the acquaintance with the commutative property of products, the deduction of the rules for the comparison of numbers by subtraction and division to pupils of the second grade; the deduction of the rule of multiplication of numbers by 10, 100, and so forth, to pupils of the third grade. Such cases are encountered particularly frequently in the fourth grade, in which one of the fundamental problems of the course is the generalization and systematization of all the factual material which relates to the properties of arithmetical operations, to the relationships among the various operations and the applications of each of them, and others, which have been accumulated by the children in the preceding years of instruction.

The third case is not so often encountered in practice. However, it too is of a certain interest, precisely because although here the element of the children's independent "discovery" of new facts and modes of operation is predetermined by the teacher's preliminary work, the children do not realize this and remain fully convinced that they made the "discovery" completely independently.

As an example of the use of practical operations with objects as a starting point in the explanation of new arithmetic material, we may take the practical work in the comparison of the lengths of two strips of paper (ribbon, string) immediately before the examination of the first problems on the comparison of numbers by subtraction.

As for independent work with a book, only one of the instances of such work carried out by a teacher (A. B. Kozokina) in the fourth grade was described above. However, in the trial instruction in experimental classes we succeeded in carrying out such work not only in the fourth, but in all the other grades, beginning with the first. Thus, after having repeatedly gone over with the children the detailed form for writing down the solutions of various examples, indicating the chain of reasoning of their solutions, we finally used the following assignment as material for independent work:

> Examine the way the solution of one or another new example is written down in the book and be prepared to explain its solution according to the way it is written down.

In many cases, the method of writing the solution illustrated in the book was given to the children as a model from which they had to write down independently, the solution of another, structurally analogous example. For the organization of independent work using a book in the second and third grades, we used the illustrations given in the book of the conditions of several problems and others.

In all the cases which do not introduce any material new in principle, very intensive preparation is necessary so that independent work on new educational material will be within the children's capacities. The first and necessary condition for this is the children's conscious and firm mastery of the knowledge upon whose use the solution of the new problem is based. For this reason, such independent work may be carried out in class only after the teacher is convinced of such mastery. All the necessary facts, rules, definitions, etc., absolutely must be reviewed immediately before the execution of the independent work.

Observation further convinced us that in order to prepare for the execution of independent work, the necessary problems from past experience must be repeated separately from the others and must be divided into parts so that in giving the assignment the teacher can say:

> I see that you know so and so and so and so, know how to do so and so and so and so, thus you should be able to solve this new problem (or answer this question) by yourselves.

This form of assignment, more than anything else, serves to activate in the children's consciousness precisely the knowledge which they must use to fulfill the assignment.

Furthermore, if the knowledge acquired earlier must be applied by the children under conditions rather different from those they have encountered earlier, it often proves useful and even necessary to pay special attention to what in these new conditions is similar and what is different in relation to what is known. For example, in the lesson devoted to addition, without carrying over ten but totalling less than 100, before asking second grade children to independently solve the example 32 + 6 we not only reminded them of the familiar case of addition without carrying over ten totalling less than 20 (12 + 6), but also asked them to compare these cases. This comparison was made under the teacher's guidance. The children reproduced the whole chain of reasoning with a familiar example, then the difference between it and the new one was formulated. ("There there was 1 ten, and here there are 3 tens.") Then the teacher said:

> So, children, this is the only difference and in other respects these examples are very similar. Now solve the new example by yourselves, reasoning as you did earlier, and be prepared to explain the solution.

After the explanation of the solution to this example, other examples of this type were assigned for independent solution, and the children confidently fulfilled this assignment.

Then the children went over the appropriate reasoning aloud during group work in the next lessons. Having acquainted ourselves with the "exercises with commentary" which were used in the experimental work of the Lipetsk teachers, we came to the conclusion that it is precisely at this stage, during the consolidation of the knowledge of the rule for solving examples of a new sort, that they should prove to be particularly useful, provided that the pupils' "commentaries" during the solution of examples are gradually curtailed from one lesson to the next and then completely disappear.

Thus, in the case considered above of addition without carrying over ten but totalling less than 100, in the first lessons the explanation given during the solving of new examples of this type would, for example, sound like this:

> Add 6 to 32. The number 32 consists of 3 tens and 2 units. We will keep the 3 tens in mind and add 6 units to the 2 units, making 8 units; and there are also 3 tens—30 units—added to 8 units—makes 38 units in all.

After a while (depending on the children's mastery of the rule being studied) the explanations will be formulated considerably more briefly (using the same example): "Add six to 32. Two added to six makes eight; eight added to 30 makes 38." After the children have mastered the method for subtracting there is no longer any necessity for such explanations during the solution and the pupils are granted the right to say at once what six added to 32 equals.

However, from time to time (especially if one or another error appears in subtraction) the teacher will ask how the subtraction was carried out. Each pupil should at any moment be able to give a detailed explanation of the whole process of solution.

Thus, the deduction is made on the basis of the children's independent examination of a new case during the same lesson at which it is introduced, and is consolidated during the next lessons under the teacher's guidance and control. The teacher's guidance here must be directed toward the children's conscious mastery of deduced rule and the general automatization of the operations dictated by it.

In general, pupils' independent work in all cases can be considered only as a starting point in the study of new material. It not only does not exclude, but absolutely requires, the teacher's well-thought-out guidance at each stage of the work on the new material.

It is not out of place to recall here an extremely important conclusion, in our view which was drawn on the basis of a psychological investigation of the process of mastery of knowledge in school children. This investigation concluded that the first acquaintance with a new concept, rule, etc., however it is organized (on the basis of the teacher's explanation or on the basis of the children's independent work)

> is only an initial phase, a point of departure for mastery: the further fate of this process depends on how the concepts, rules, etc., are organized in instruction. During the exercises, knowledge is not only consolidated, but also extended and made more precise [2].

For this reason it is not possible to consider that the organization of children's independent work at the stage of the initial introduction of one or another new problem indicates that the children have independently mastered the appropriate knowledge. Even at this moment, their activity proceeds under conditions created by the teacher, prepared and directed by means of the materials, exercises, questions, etc., that she has selected. The teacher's guidance fully retains all its significance in the next steps of the acquisition of new knowledge--during the children's fulfillment of various assignments connected with the application of this knowledge.

To summarize what has been said in this chapter about the place of pupils' independent work in arithmetic lessons in the elementary grades, it is possible to formulate the following conclusions.

1. Pupils' independent work can and must find a plce in every arithmetic lesson in all grades, beginning with the first.

2. In most lessons it proves expedient to organize the children's independent work along the lines of various assignments by the teacher, more than once in the course of the lesson (two to six times in one lesson), depending on the goals and problems of each lesson, the peculiarities of the educational material on which the independent work is constructed, and the children's level of preparation.

3. Pupils' independent work can be used with success at various stages of a lesson, including the introduction of new educational material. In this chapter we considered the concrete cases in which such an organization of the work on new educational material proves productive.

The possibility and necessity of children's independent work at the stage of the consolidation of knowledge, skills, and habits which have been acquired before is unquestionable--in this stage of the work on new material it is only important to note what types of tasks can best be used here, and how the appropriate independent work must be carried out in the lesson. The next chapters are devoted to the consideration of these problems.

Assignments for Pupils' Independent Work with Various Educational Materials

It is possible to find, in the methodological literature, assignments diverse in content and character which are used by individual teachers in instructing children in arithmetic in the lower grades of school. However, our observations indicated that far from all of these assignments have a practical application in the experience of the mass school.

Indeed, if the tables reflecting the general picture of the organization of pupils' independent work in the lessons we observed are reviewed, it is apparent that the number of kinds of independent work represented is extremely limited. Even in the Lipetsk experiment (see Table 2), as a whole, the impression left by the independent work observed is one of extreme monotony. Furthermore it is impossible not to note that too much lesson time was spent on the exercises which least required that the children manifest independence.

From Table 1 it is evident that copying from the board took up approximately 18% of the time devoted to the children's independent work, and that solving prepared examples took up 41%. Thus, the greater part of the time in the lesson when the children were occupied with independent work was spent on exercises devoid of the element of creative work--the independent search for a way to solve some new problem.

Only in 30 out of 100 cases was the children's independent work in some way connected with the solution of problems—even though, as is well known, the solution of problems presents one of the greatest difficulties in the teaching of arithmetic. It is also important to note that of these 30 cases, 15 consisted only in part of independent solutions—as a rule, only the writing down of the solution to the problem which had already been analyzed from beginning to end collectively with the teacher's help. The observations indicate that even in small schools where, by force of circumstance, the teacher must keep the pupils occupied with independent work for 20-25 minutes per lesson, this time is also almost exclusively devoted to the solution of examples and writing down problems which have already been solved. During the discussions of this circumstance which we conducted at the teachers' committee on methods, it became clear that the reason for this selection of exercises for pupils' independent work was uncertainty that the children could handle the task.

> You give the assignment of solving examples independently confidently, knowing that the children as a group can handle the work, but if you ask them to solve a problem independently (even of a familiar type), many pupils will sit idle because they do not know how to approach the solution.

This is a characteristic explanation given by the teachers. However, it is hardly possible to accept this explanation as sufficiently convincing. It is impossible, in fact, to become reconciled with the position that pupils do not know how to approach the solution of even a familiar problem. Ultimately, the goal of instruction in arithmetic will not be achieved if the teacher considers this position normal and orients her instruction toward it. It seems extremely improbable that the pupil could learn to solve problems independently if he has no practice in doing so. To outline the content of such work, to select the best forms of assignment, to work out a system of these assignments so that they gradually increase in difficulty, has become one of our main tasks.

During our attendance at the lessons we observed, further characteristics of the work on examples came to light. In the lessons of all the experienced teachers with whom we were acquainted, in the overwhelming majority of cases, the assignment consisted of the written solution of prepared examples by the students. There were no essential differences in the form of the assignment--the teacher either simply indicated the number of the appropriate exercise from the textbook, wrote examples on the board, or distributed cards with examples written on them to the children (most frequently the assignment was given in two variants so that pupils sitting next to each other worked on different examples). Several teachers gave supplementary assignments to pupils who had finished their work before the others. While the children solved the examples, the teacher walked up and down the aisles, showing individual pupils their errors and helping them. As a rule, the teacher checked the work after the lesson was over.

The assignments were conducted in the same way in all classes. The only difference lay in the numbers the children dealt with. From discussions with the teachers about the reasons for this monotony in work with numbers, we became convinced that many of the teachers who limited the pupils' independent work to the written solution of prepared examples do not proceed from any special consideration, and even lose sight of the possibility and advisability of introducing variety in the work. Only insufficient attention to this can explain why, during the execution of independent work, the teachers rarely used even the kind of exercises which are relatively richly presented (with respect to variety but not quantity) in the textbooks by A. S. Pchelko and G. B. Polyak.

Taking this into account, we shall consider below the various types of tasks for pupils' independent work on the solution of independently composed and modified examples. Of these, we shall principally consider exercises of a creative nature which favor the development of children's powers of thought, observation and "mathematical insight."

Assignments for Pupils' Independent Work on Arithmetical Examples

Exercises in the solution of examples can serve the most varied purposes in the teaching of arithmetic. The most important of these purposes is, of course, the development of the skills of mental and written calculation. However, the examples can also be used for the formulation of a whole series of arithmetical concepts, for the explanation of the properties of numbers and arithmetical operations, etc. Work on examples can and should be not only highly useful in an instructive sense but also absorbing, interesting, and useful in an educational sense.

There exist varied exercises connected with the solution of arithmetical examples, each of which require from pupils strenuous thought, attention, and the ability to apply practically the theoretical knowledge they have deduced. We shall consider all possible exercises of this sort and attempt to show the purpose of each, and when and how it is best to use them.

First, let us note that even in work on the solution of prepared examples considerable variation can be introduced, both in the form of the assignments and the writing of solutions, and in the essence of the work to be done. Thus, the examples certainly need not always be given to the pupils written out in full—often it is possible to use the table for mental calculation which each school has for independent class work. For example, the teacher may give the children the assignment: "Add the numbers in the second column to the numbers in the first column, write down the examples and solve them." To give the children such an assignment, the teacher need not spend time writing the examples on the board. Such an assignment is useful for the children since it demands greater attention from them than does the copying of prepared examples from the board or a book.

Assignments of the following type are also helpful. A row of n numbers is written on the board and the children are asked to increase (or decrease) each one of them by several units or several times. For example, in the second and third grades the following assignment may be given.

Decrease six times: 36, 72, 12, 84, 96.

For the first grade:

Increase by seven: 11, 13, 6, 9, 12, 8.

Such examples allow work on the development of computational skills to be combined with the formation of very important arithmetical concepts. Thus, apart from addition and subtraction by several units or several times, the comparison of numbers both by subtraction and by

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division can be included in such an assignment. For example, two rows of numbers are written on the board.

53 48 32 54
27 19 8 12

The children are asked to compute (and write under each pair of numbers) by how many units the numbers of one row are greater (or less) than the numbers of the other row.

This kind of independent work for pupils can sometimes take the place, in a lesson, of so-called mental calculation. It is wise to limit beforehand the time spent on an assignment; in this case the children can solve the examples orally and rapidly. This form of work allows the teacher to check on how each pupil is handling the assignment (this is certainly not always possible to accomplish during questioning of the class in mental calculation). For the teacher who works with two or more classes at a time this type of assignment is especially good, because the lessons on mental calculation can be conducted without the teacher's direct participation in the work. At present, according to our observations, this form of assignment is used, if at all, during the conduction of practice in mental calculation under the teacher's guidance. Furthermore, even when the examples are given to the children in written form, it is not at all necessary that they all rewrite them. In many cases, the writing can be limited to just the answers.

To carry out such work in the third and fourth grades, the teacher must have a set of cards on which the assigned examples are already written in columns. For example:

$$\begin{array}{rrrl} 13567 & 3541 & 348 & \\ +\ \underline{26782} & -\underline{2762} & \times\ \underline{\ \ 3} & \text{etc.} \end{array}$$

If a number is indicated on each of these cards, then the pupil writes it down in his notebook and may solve the examples without rewriting them. Or even better, if all the examples are arranged in one row, then the pupil may align the sheet he has received with a page in his notebook and, after solving each example, write just the answer under it.

It is useful to introduce an element of self-guidance into work on the solution of prepared examples. This element arouses the pupils' interest and, moreover, increases their feeling of responsibility for the work they have done. The well known "circular" examples may be used to introduce such self-guidance. During the solution of these examples, the children continually check, to some degree, the correctness of each answer they have gotten. Unfortunately, even "circular" examples are seldom used by the teachers as material for children's independent work.

It is possible, in assigning to the children several examples for solution, to indicate, let us say, that the sum of all the answers must equal 53 (or some other number). Assume the following examples were assigned:

$52 - 46 = \ldots;\quad 13 + 9 = \ldots;\quad 72 - 67 = \ldots;\quad 18 + 2 = \ldots$

After having solved them and arrived at the proper answers:

6, 22, 5, 20

the pupil must add these numbers as a check on computation. If the result he gets is not 53, then he must again check the solution of each example. Practical experience indicates that pupils, until the upper grades, do not know how to check their arithmetical work. In teaching them, special attention must be paid to the checking of computations when independent work is being carried out.

Along with the introduction of elements of self-guidance into the assignment itself, and also with the reciprocal checking which many teachers practice, it is possible to use the following type of work. The children are given the assignment of checking the solution of examples written on the board (or on a card) and writing in their notebooks the correct solution of only the examples in which there was an error.

Pupils also apprehend with great interest assignments to write out examples from given answers. This procedure also requires a combination of mental calculation with writing. Several variations of this assignment are possible. Let us take, for example, a simple case in which the teacher gives the children a row of examples and the answer 6, and asks them to write in their notebooks the solution of only the examples which have the answer 6. In a more complicated but more interesting form of the task, the teacher makes up a row of such examples to which the answers are, for example; 11, 12, 13, 14 etc. These examples are written down pell mell and the children are given the assignment of writing out first the example with 11 for an answer, then with 12, then 13 etc. For example, the teacher may make up examples on division beyond the table but within the bounds of 100:

44 ÷ 4 = 11   91 ÷ 7 = 13   75 ÷ 5 = 15   68 ÷ 4 = 17
72 ÷ 6 = 12   42 ÷ 3 = 14   48 ÷ 3 = 16   36 ÷ 2 = 18

These examples will be given to the children, let us say, in the following sequence:

48 ÷ 3 =   75 ÷ 5 =   36 ÷ 2 =   68 ÷ 4 =
72 ÷ 6 =   44 ÷ 4 =   91 ÷ 7 =   42 ÷ 3 =

In the execution of this assignment the children rename the quotients mentally many times. In the search for the first example—with 11 for an answer--they rename four quotients; in the search for the second--with 12 for an answer they return again to the examples they have just renamed and rename the first two. Then, looking for the example with 13 as an answer, they must rename the first 6 quotients, etc. The assignment can be completed more quickly if the pupil remembers the results which he got through mental calculation. Thus, the teacher can use this sort of task for the organization of a contest, based on speed of solution.

Finally, the solution of prepared examples may be facilitated by casting them into the form of a game of lotto. Arithmetic lotto can be made to suit the curriculum of each class and can be successfully used both in and out of class.

It is a small thing, however, merely to vary the tasks connected with the solution of prepared problems. It is more important to turn to assignments which would, more or less, demand creativity from the children which would cultivate their powers of observation and independent thought. Here I have in mind the most diverse tasks for pupils on the supplementation and independent construction of examples. The simplest such task is the solution of examples with blanks. One of the addends, one of the multiplicands, or a sign of operation, etc., is omitted. Here are instances of this sort of example:

54___ 9 = 63, 24 ___ 3 = 21, 36 ÷ ___ = 12, 18 + ___ = 27

Examples with operation signs omitted are most suitable for the first and second grades, because they make the children pay attention to the signs for arithmetical operations, which children of the lower grades tend, at times, not to notice. However, this task, if it is made more complex, can be used with benefit in the third and fourth grade. Pupils can be asked to fill in the operation signs in such examples as: 37 ___ 8 ___ 7 = 38, 56 ___ 12 ___ 14 = 54 etc., or in still more complex examples which require a firm knowledge of the order of operations: 48 ___ 3 ___ 5 = 33, 36 ___ 12 ___ 4 = 33 etc.

Examples which require the discovery of the second addend from the sum and first addend or the discovery of the multiplier from the product and multiplicand, or the discovery of the minuend from the subtrahend and remainder, etc., can be given successfully to all grades. In the first and second grades it is best to give them in written form. The assignment itself is formulated as follows:

> Copy these examples and fill in the blanks. For example:
>
> ___ + 2 = 8, ___ x 6 = 24, 17 - ___ = 4 etc.

In the third and fourth grades such examples can also be given, but here the form of the assignment can be varied. If the students of these grades are familiar with the names of the components of the arithmetical operations, this knowledge can be used and examples of the following type assigned. "Multiplicand--136, product--1088, find the multiplier." Another interesting exercise for third-and fourth-grade pupils is filling in the blanks in examples of the arithmetical rebus. For example:

```
   _3_65       54_
 + 7_36_       x 6
 -------      ----      etc.
   828_0      __94
```

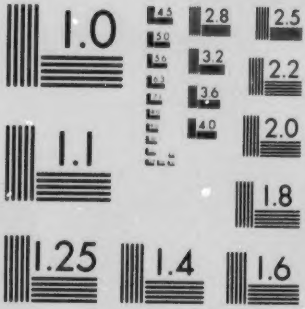
This is not an easy task, and in order to complete it the pupils will have to utilize much of the knowledge which they have acquired at various times and during the study of various problems of the curriculum.

The varied cases of constructing examples are exercises of a creative nature. Let us consider tasks of this sort. The teacher can ask the children to construct five or six multiplication examples, four examples in which it is necessary to add six, etc., when the operation and one of the components are given. Exercises of this sort are especially useful for the first and second grades.

Great benefit is obtained from the construction of examples from a given operation and answer. For example; "Make up addition examples with 10 for an answer" or "Make up multiplication examples with 48 for an answer." In this case, the difficulty of the task depends on whether or not it is precisely indicated how many examples must be made up. If the teacher does not specify the number of examples and ask them to construct as many of these examples as possible, the children's task becomes more difficult but also more interesting. The construction and solution of such examples promotes the children's better mastery of the composition of numbers being studied, both of addends and of multiplicands.

Sometimes it is possible to ask the children to construct several examples using a particular operation without any additional limitation or with a given answer. In the latter case the children can construct examples using all the arithmetical operations they know in any combination. In this way, from a single answer, it is always possible to construct many examples. For this reason, it is necessary to put a definite time limit on the task. Indeed, even if in the first grade, after the study of the first ten numbers, the children were asked to construct all possible examples with five for an answer, many such examples can be composed: 4 + 1; 3 + 2; 2 + 2 + 1; 3 + 1 + 1; 6 - 1; 7 - 2, and in addition, examples in which both addition and subtraction are used.

Analogous work can be quite difficult and useful even for fourth grade pupils. Thus, to make up examples in division from a given quotient, the pupil must apply the knowledge which he possesses, under completely new circumstances. Experience shows that even if children have earlier practice checking division through multiplication more than once, finding a dividend from the divisor and quotient still causes difficulty for many pupils.


1.0
4.5
5.0
5.6
6.3
7.1
8.0
2.8
3.2
3.6
4.0
2.5
2.2
2.0
1.1
1.8
1.25
1.4
1.6

MICROCOPY RESOLUTION TEST CHART

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The construction of examples from three numbers is not difficult and, moreover, it is very useful. For example, the numbers 3, 4 and 7 are given. The children must construct all possible examples, namely: 3 + 4 = 7, 4 + 3 = 7, 7 - 4 = 3, 7 - 3 = 4. Or the numbers 2, 6 and 12 are given and examples constructed from them: 2 x 6 = 12, 6 x 2 = 12, 12 ÷ 2 = 6, 12 ÷ 6 = 2.

Such exercises are conducted in the first and second grades. Much before they consider the connection between addition and subtraction or between multiplication and division, in theory, the children are introduced practically to these connections through these exercises. The accumulation of such experience greatly helps the pupils in the upper grades, where, on this basis, they arrive at the appropriate generalizations. In the third and fourth grades, with this same purpose in mind, it is very useful to require the children to check the examples they have solved by various means, including the use of the reverse operation.

The following task, which fosters "mathematical insight" in children, is also interesting.

> Construct as many examples as possible using any operation and the given numbers (For example: 132, 75, 11, 144, 3, 24, 15).

The pupils can construct the following division examples:

132 ÷ 11 = 12   75 ÷ 3 = 25   144 ÷ 3 = 48   24 ÷ 3 = 8
132 ÷ 3 = 44   75 ÷ 15 = 5   144 ÷ 24 = 6   15 ÷ 3 = 5

It can be indicated in the instruction the number of examples to be constructed. This indication facilitates the children's task, as they will strive to find the possible variants if all the eight cases have not been exhausted. Numbers for the composition of other examples can be selected analogously.

Finally, the following assignment on the construction of examples of a definite type is possible.

> Construct eight examples of addition of one-digit numbers, carrying over 10; five examples of subtraction of many-digit numbers; or several examples of division with a zero in the quotient, etc.

In this case too, the pupil is required to apply earlier-acquired knowledge under new conditions which promotes a deeper awareness in the children of what they are studying.

Pupils' construction of examples in varied assignments can be used to create supplementary material to be distributed to the class, a procedure which later helps to organize the children's independent work during the lesson. For example, the children may be asked to construct six examples of addition, carrying over 10, totalling less than 100. The teacher explains that the work must be done on separate sheets, that all the examples must be written one under the other in a column and that the answers must be written four or five squares away from the equal sign. For example:

$$24 + 8 = \quad 32$$
$$37 + 9 = \quad 46$$
$$58 + 5 = \quad 63$$

In the third and fourth grades, in which the pupils must practice written calculation, the assignment may be given to construct three examples of the addition of many-digit numbers; in each example there must be three addends. These examples may look as follows:

$$\begin{array}{r} 3786 \\ 542 \\ \underline{+\ 5618} \end{array} \qquad \begin{array}{r} 5423 \\ 1496 \\ \underline{+\ 358} \end{array} \qquad \begin{array}{r} 79628 \\ 754 \\ \underline{+\ 85219} \end{array}$$

The answers are to be written several spaces below the lines.

After the teacher has checked the work, he cuts the answers off but keeps the sheets with the examples. Later the examples can be used as cards for individual work with several pupils (during a question period, or during the class period for independent work). Exercises which require the children to analyze, compare and notice the regularities in observed arithmetical facts are also very useful. Such assignments can be both very simple and very complex. For example, even in the first grade the children can be asked to continue the series of numbers 1, 3, 5... or 2, 4, 6.... In the second grade, when the pupils study the table of multiplication and division within the bounds of 100, they must be able to note the principle of construction of each table and to continue independently, writing the table which was begun under the teacher's guidance. Here they can be asked to complete series like 2, 4, 8... or 3, 6, 9 ....

In the third and fourth grades assignments of the following sort can best be used.

> Continue the given series of examples according to the same principle:
>
> 124 ÷ 4 $\qquad$ 1 x 9 + 2
> 224 ÷ 4 $\qquad$ 12 x 9 + 3
> 324 ÷ 4 $\qquad$ or $\qquad$ 123 x 9 + 4

The assignments considered above for pupils' independent work on the solution and construction of examples do not exhaust all the possible variants. But it seems to us that there are enough of them to indicate how diverse, interesting and, above all, how beneficial for the children the minutes of independent work at a lesson can be, if this work is not reduced to the execution of uniform assignments and the solution of prepared examples.

Assignments for Independent Work Directed Toward Instructing Children In Problem Solving

It has already been noted above that the extent of pupils' independence in the solution of problems is, as a rule, in practice very small. The course of problem solving is usually analyzed collectively. The teacher helps the children realize the conditions of the problem, help the children discover the relation between the unknown quantity and what is given and uses diverse methodological devices to facilitate the search for the means of solution through dramatization of the problem or by posing of leading questions of the type: "Can we answer the problem's question at once? What else must we know in order to do so? Can we find this out from the conditions of the problem?" or "What can be found from these given quantities? Does this help us answer the problem's question? What can we find out then?" etc.

Often, under such direct guidance by the teacher and with his direct help, the whole solution to the problem, from beginning to end, is analyzed. Sometimes it is even written on the board so that the pupils need only rewrite the prepared solution in their notebooks.

Devices which make greater demands on the pupils are considered to be those where, for example, the problem is solved on the board and then erased or covered after which the children are asked to write the solution independently, or those where only the results of each operation are written on the board. However, these variations do not alter the main point--in all these cases the pupils can reproduce the solution from memory. But in none of these instances do the pupils show any independence in the actual solution. It is true that teachers also use assignments in which the children must complete a part of the task with real independence. But these assignments are usually given only in cases where the children solve a problem which they have already studied, or an aspect or problem well known to them, analogous to one which has just been analyzed under the teacher's guidance. Thus, when they complete the solution to a problem or solve it independently, the pupils, instead of trying to grasp the conditions of the problem and looking for the way to solve it, often try to recollect the operations as they were used in earlier analogous cases and to recall the means of solution to problems of this sort.

Practice shows the consequence of the solution of problems "by analogy" in accordance with a developed formula. After a short period devoted to the study of a particular new type of problem, the pupils appear to have learned how to solve it, but after they have been introduced to two or three new types of problems, the children begin to have difficulty in solving ones which they formerly managed easily. "I've forgotten these problems," "I've confused this problem with another one," the pupils often say to justify the errors in their solution. They proceed from the assumption that it is necessary to remember the solution of problems of various sorts.

Experience convinces us that a large number of repetitions and reproductions of the means of solution to a problem of one or another type, on the whole, does not equip children with the ability to independently analyze the conditions of a new problem (even an easier one) or to find the way to solve it. Moreover, the methodology of instruction in solving problems, in particular the methods of conducting children's independent work on problem solving, is built, at present, mainly on this principle—listen to the course of a problem's solution, understand it, learn to apply it in analogous cases. Another principle must be juxtaposed to this one—first learn to read the text of a problem (any problem, not just a familiar kind), learn to single out the question in the text of the problem, learn to pose questions and use devices which help to uncover the relation between the given quantities and the unknown, and learn to apply those devices to the solution of every problem whether it is of a familiar or an unfamiliar type.

This principle for approaching the teaching of solving arithmetical problems is formulated in one of the latest works devoted to the psychology of teaching school children [3: Chapter 5]. It is also reflected in the explanatory notes to the arithmetic curriculum where it is emphasized that in order to teach the pupils to independently solve all the arithmetical problems which are within their capabilities, one must remember that

> teaching several general ways to approach the solution is essential...pupils must learn to read correctly and with comprehension the conditions of the problem, to briefly and clearly write down the conditions of the problem, to illustrate the conditions with the help of a drawing or diagram... to apply several abstract terms "price," "quantity," "value" and others [4:51-52].

We attempted to give some specificity to this principle in considering various types of assignments for children's independent work on problems. Below we consider concrete assignments involving various stages of work on problems.

Independent tasks involving the perception and analysis of the condition of a problem. The conscious perception of the conditions of a problem, the ability to clearly visualize what is discussed in the problem, the ability to single out its most essential conditions and determine the relationship between the various given quantities, and also between the given quantities and the question—is the first and necessary step in the solution.

It is well known what great assistance in the search for the means of solution can be rendered by such devices as briefly writing down the conditions, representing them diagramatically and others. In the methodological handbooks it is emphasized more than once that to form proper skills and habits it is necessary to use, along with the teacher's explanation and work carried on with his indirect participation, children's independent work. The following types of assignments are especially directed towards formation of the skills referred to above.

The first group of assignments is related to the development of the ability to correctly read and understand the conditions of a problem. The assignment, "Read by yourselves problem no. ___" (or the problem written on the board by the teacher), should be heard considerably more often in class than it is at present. A propos of this, we note that when asking the children to read the conditions, it is necessary to give them enough time to do so (it often happens that the teacher, after giving this assignment, immediately asks one of the pupils to read the problem aloud or does so himself). However, the assignment "Read the problem yourselves" does not always stimulate the pupils' active work. Frequently the children do not start to read because they assume that the conditions of the problem will be read aloud anyway.

It is more expedient to give assignments in which the pupil must learn something from his reading, as when the teacher says:

> Read carefully to yourselves problem no. __; be prepared to read its question aloud.

In checking the completion of this task, the teacher, in this case, watches to see that the pupils read only the answer; thus the pupils are really obliged to prepare an answer beforehand. Such an assignment, moreover, requires the pupil to conduct a preliminary analysis of the problem's text. In the assignment under consideration such variations as the following are possible:

> Read the problem and be prepared to tell what the number 5 signifies and what is known in this problem.

Teaching the independent reading of problems should begin with the first grade and continue into the fourth grade. The difficulty of the assignment to be worked upon independently in this case will increase as much with the additional complexity of the problems to be solved as with the greater difficulty of the assignment, in regard to the analysis of the conditions. Although in the first and second grades it is possible to limit the assignment, for the most part, to tasks of the sort described above, in the third and fourth grades the demands will be useful and more complicated--for example,

> Read the problem and single out pairs of the given quantities which are related to each other.

As a further example, consider problem no. 216 from the fourth grade textbook [16]:

> On a collective farm there are 108 calves and 65 cows more than that. Each calf is provided with 12 centners of silage for the winter and each cow with four times as much. How much silage in all is provided for the winter for the calves and cows?

By reading and rereading the problem, the children single out pairs of given quantities, and name them when the teacher asks.

1. On a collective farm there are 108 calves and 65 cows more than that.

2. On a collective farm there are 108 calves. Each calf is provided with 12 centners of silage for the winter.

3. Each calf is provided for the winter with 12 centners of silage and each cow with four times as much.

Such preliminary independent work on the conditions of a problem can often form a useful starting point for its solution.

The second group of assignments involves the sketching of conditions. The sketching of the conditions of a problem often helps the children understand it better and participate more actively in further individual or collective work on the analysis of the solution.

The assignment "Sketch the conditions of a problem" can be very easy or relatively difficult depending on the characteristics of the problem itself. Thus, it is easy for instance, to sketch the conditions of a problem like no. 76 from the first grade textbook.

> Vitya cut from material three black circles and four green ones and made a penwiper out of them. How many circles in all were used for the penwiper?

To illustrate the problem, the children represent the objects which the problem mentions. The next step is when the children, to illustrate the conditions of a problem, replace the objects which are mentioned in it with others. For example, in illustrating the conditions of problem no. 413:

> On a Christmas tree there burned three green lamps and six red ones more than that. How many lamps in all burned on the tree?

The children can replace the lamps with circles. In other cases, instead of drawing boys, girls, etc. they can draw the proper number of sticks, boxes, etc.

Finally, even in the first grade the children can begin to use some of the designations for conditions. Thus, in illustrating problems on finding a remainder, for example, they can be taught to use the device of crossing out--a device which the authors of the textbook continually use in the consideration of various cases of subtraction. Let us take as an example the solution of problem no. 159:

> Vasya must cut out nine stars. He has cut out eight stars. How many more stars must he cut out?

To illustrate the conditions of the problem, the children draw nine stars and then cross out eight of them.

However, sketching the conditions of problems in the first grade must not be misused--in many cases it is simply unnecessary. Thus, in the solution of the simplest problems in finding remainders and sums, it is useful only the first few times when these exercises are solved completely on the basis of objects as visual aids. But after the children go over to the solution of such problems by ideas, using the methods of addition (or subtraction), the sketching of the conditions is a superfluous and even a harmful task, because it returns the children to an earlier stage. Later, in introducing new problems, first the increasing (or decreasing) of a number by several units, and then on multiplication and division, it is again useful at the first stage of the work to use sketching, since it makes it possible to add precision to the children's understanding of the problem's meaning, the meaning of the operation of multiplication, and its relation to addition.

Assignments which require the independent sketching of the conditions of a problem are useful not only in the first, but also in the second grade. They are of the greatest interest when the solution of a problem is based on the exact understanding of expressions like "as much as," "so much more than," "so many times more (less)," because the correctness of the drawing predetermines the success of the solution. The teacher, by checking how the children sketched the conditions, and by being convinced by their drawings that they understood the problem, can confidently ask the pupils to complete the solution independently.

It is also expedient to resort to the illustration of conditions in introducing several new types of problem in the third and fourth grades. However, simple sketching must gradually be replaced by the diagrammatic representation of a problem's conditions, reflecting the connection between the quantities given in the problem and the unknown.

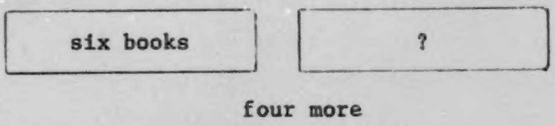
The assignment "Sketch the conditions of a problem" can be given as an independent task after the problem is read aloud by the teacher or one of the pupils. The same problem can be given, let us say, for solution at home. But its solution also can be analyzed in class under the teacher's guidance--it all depends on the characteristics of the problem itself. However, it soon becomes possible to combine this assignment with the children's independent reading of the conditions.

The third group of assignments involves outlining the problem's conditions. This outline, with the use of several designations for the conditions--diagrammatic representation, graphs, tables, etc.—naturally can be assigned for independent work only if the pupils have become familiar with these devices under the teacher's guidance and have learned to use them. It is not worthwhile, however, to delay the use of assignments of this type too long. In an explanatory note to the arithmetic curriculum it says that "in the third grade the pupils must...be able to illustrate, diagrammatically, the conditions of a problem." Consequently, it is sometimes thought that before third grade is too early to introduce children to the illustration of problems. However, the stated goal can only be achieved if the children are systematically and regularly taught the devices for the diagrammatic representation of the conditions of a problem or the outlining of them. Experience has convinced us that this work can be successfully carried out starting from the first grade.

Thus, in the first grade, when teaching the solution of problems on increasing (or decreasing) a number by several units, the children were familiarized with a diagrammatic outline of the conditions. For example, consider problem no. 269.

> Petya has six books and Mitya has four more than that. How many books does Mitya have?

The conditions are outlined as follows:



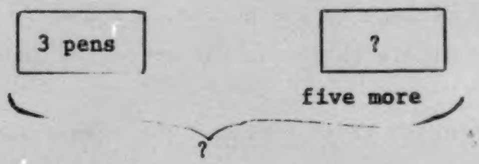
This diagram facilitates the analysis of the problem, the clarification of what is known and what is not, and helps define the type of problem. Moreover, the use of such diagrams facilitates the organization of the children's independent work on the solution and independent construction of this type of problem. Indeed, when the children have been instructed in ways to represent the conditions of a problem, the teacher can, let us say, put two similar diagrams on the board and ask the children to independently imagine problems for the diagrams and solve them. Or, the teacher can read a problem and ask the children to construct a diagram for it in class and then solve it at home.

At first, of course these diagrams are drawn by the teacher. But the teacher should gradually encourage individual pupils to construct their own diagrams after which the appropriate work is carried out by all the pupils under his guidance. After several exercises in the construction of such diagrams, the children can complete the appropriate assignments independently.

The ability to represent the conditions of a problem significantly facilitates children's understanding and their search for the method of solution when they deal later with compound problems containing simple problems of this sort. Indeed, if the children have really mastered the work described above, they will not have great difficulty even with the diagrammatic writing of the conditions of a problem in two operations. Consider, for instance, the problem:

> In one box there are three pens, in another five more than that. How many pens are there in the two boxes? (No. 411)

The diagrammatic outline of the conditions will, in this case, look as follows:



The two question marks in the diagram alert the pupil attempting to solve the problem. Those pupils who have really learned to read such an outline do not stop with the solution of the first operation (a typical mistake). In the first grade the dimensions of the rectangles in such diagrams must not reflect the quantitative relationship with which the problem is concerned, since it is already difficult for first graders to realize that these rectangles have somehow replaced the most diverse objects--boxes, pails, pieces of material, etc.

We have dwelt in detail in considering a particular concrete instance of the use of a diagrammatic outline of the conditions of a problem in order to show that even the material of the first year of instruction urgently requires the use of this device. In the following grades, when more and more complex problems are studied, the use of a diagrammatic outline acquires even more significance in clarifying conditions.

It is important that not only the teacher, but also the pupils, understand that the outline of the conditions, or one or another illustration of them, is not an end in itself--but rather they are only a means, making the content of a problem easier to understand. The question of what form of outline is better depends on the characteristics of each concrete problem. Most frequently the teacher indicates the form directly e.g.,: "Read the conditions of this problem and represent them with the help of segments," "Make a sketch of this problem," or "Write down the conditions of this problem, using a table," etc. However, in a number of cases it is useful to leave the choice of the most suitable form of representation to the pupils themselves.

The tasks considered above, in many cases, entail an analysis of the conditions of the problem, since to construct a correct diagram, for example, it is necessary not only to recognize the conditions, but also to discover the relationship between the unknown and the given data. The construction of a plan of solution and the solution itself present a whole series of special demands to the pupils, and the children must gain mastery of the appropriate skills in the course of independent work.

The partially independent solution of a problem. The transitional stage between the analysis of problems under the teacher's direct guidance and the completely independent solution is the partially independent solution, which entails the teacher's help at a definite stage in the work. Thus, the teacher can ask the children to solve a problem independently after, under his guidance, the conditions are repeated, the type of problem is defined, and certain preliminary remarks are made which fix the children's attention on the most difficult moment in the solution. For example, consider the solution of problem no. 794 from the third-grade textbook:

> On one plot there are 10,820 trees, on a second there are 1,976 trees fewer than on the first, and on a third, half as many as on the first two combined. How many trees in all are there on the three plots?

If the teacher thinks that the class is insufficiently prepared for this problem, he can ask one of the pupils to represent its conditions graphically on the board. When the teacher is convinced that the pupils understand the conditions of the problem, he can ask them to solve it independently.

If such problems do not cause the pupils of a class any particular difficulties, yet the teacher feels that the children may make errors in the solution because they did not pay sufficient attention to the instructions in the text, "half as many as the first two combined," he can say, "Before you solve it, reread what is said about the third plot," or he can ask one of the pupils to read this part aloud. If, for example, a problem on division according to content is being solved in the second grade, then after the conditions are read the teacher may ask the children the question: "What type of problem is this?" When the type of problem is determined, the children can solve it independently. The assignments outlined below require a really independent solution of the problem by the pupils, but with somewhat simplified conditions.

Forms of tasks in which the children independently complete only a part of the solution are widely used in practice. For example, an incomplete analysis of the solution of a problem may be conducted in class under the teacher's guidance and with his help where the children are asked to finish the solution independently. From the standpoint of development of the proper skills, assignments which are also possible and useful are ones in which the children need only formulate the questions or short explanations of each operation of a prepared solution, or, on the other hand, select the appropriate operations to use at each point of a previously formulated plan of solution. In order to organize such a task, the teacher should have a good selection of appropriate flash cards. The preparation of such cards may be carried out in the upper grades during the pupils' independent work. For instance, the teacher may give the pupils cards on which are printed the text of problems (either from the printed didactic materials by N. S. Popova, or from some arithmetic book). The children are asked to write out a plan for solution, but not to write the solution itself (the plan is written on a separate sheet), or, on the contrary, to write only the operations, etc. Ultimately, these sheets, together with the appropriate cards, serve as material for independent tasks of the type described above (the teacher need only take care that the pupils do not get the same problems). Several supplementary variations are possible in assignments of this type. The solution can be given in abstract numbers so that the pupils must supply the denominations. This is useful for problems in whose solutions pupils often make errors in supplying denominations (for example, in problems on division "according to content)." It is possible to tell whether the children gained an understanding of the conditions and question of the problem, and the meaning of the operations leading to its solution from the way they fulfilled the assignment.

The last type of assignment involves the construction of a plan and the completely independent solution of the problem. Such an assignment is possible only if the pupils are sufficiently prepared for it. All the types of assignments considered above which demand, at least partially, independent completion of one or another part of the solution, serve to prepare the children for completely independent solutions. However, as we noted above, exercises directed only toward the formation of particular individual skills are not enough.

Even if the pupil is well able to read the problem, illustrate it, etc., this does not guarantee that he will be able to handle its independent solution as a whole. It is important to equip the children with the ability to select, from the familiar devices and methods of approaching the analysis of a problem's conditions and solution, the ones which are most appropriate to a particular concrete problem—it is important to teach them a plan for working on a problem. The pupils must also know what they must do when they receive an assignment to solve a problem. It is necessary to teach them this specially [3].

The appropriate knowledge, as always, is acquired by the children under the teacher's guidance—at first, during the collective solution of problems with the teacher's aid, and later in the course of independent exercises. When assigning the children a problem for independent solution (in class or at home), the teacher must help them project a work plan the first few times. For example, when assigning the children some problem from the textbook as homework, the teacher may ask them:

> How will you execute the assignment: what must be done first? (First the problem must be read.)
>
> And then? (Read the whole problem again and repeat the problem's question.)
>
> And if you have not yet understood the problem very well, if you do not visualize very clearly what the problem is dealing with, what will you do? (I shall make a drawing of the problem.)

The plan for work on the problem can be recognized and mastered by the children as a result of frequent contact with it. The appropriate "rule" is far from always formulated (this is true in many of the cases which are beyond the children's comprehension), but whenever he is analyzing the problem with the children or checking the way they solve it themselves, the teacher must make sure that this rule is followed. The pupils thereby get used to working from a definite plan. In the process of instruction this plan is enriched with new elements and the "rule" for the solution of problems becomes more comprehensive. The pupils' independent work will be constructed differently, depending on how well they have mastered these "rules" (that is, how well they are able to act in accordance with them).

At first it is useful to divide the children's independent solution of a problem into separate stages, checking the work at each stage. For instance, the teacher may ask the children to solve the problem independently:

> First read the whole problem carefully and be prepared to repeat it. When you are ready, sit up straight so that I can see who has completed the assignment.

Having made sure that all the children have finished reading, the teacher then asks the children to make a sketch of the problem's conditions. After this part of the work has also been completed, the sketch is checked. Then the children are asked, for example, to write down a plan for solution; the way they handle this is also checked. Finally, the children independently write out the solution and answer to the problem. After this, perhaps, they are asked to carry out a check of the problem's solution. Each step of the work is checked collectively under the teacher's guidance. Later, the independent solution of the problem can also be carried out by division into separate stages of work, which are outlined by the teacher, but without a check of the work at each stage. Finally, an organization under which the children solve the problem independently becomes possible too, and the work is checked as a whole.

Independent tasks involving the transformation of conditions, their completion and the construction of problems. Independent tasks of the type indicated have especially great significance for the development of pupils' thought processes. The completion of a problem, the transformation of its conditions, the independent construction of a problem--all are exercises which make the children penetrate deeply into the very essence of the problem being considered, and gain an understanding of the peculiarities of its construction and the relationships between the quantities which are given in it. For precisely this reason, these exercises are among the most valuable means of teaching problem solving.

Let us consider some assignments for independent work dealing with the *transformation of a problem*. Usually, the transformation consists in the following. After the solution of a given problem, the former unknown becomes one of the given data, and one of the given data of the given problem becomes the new unknown. Such a transformation is often made under the teacher's guidance, but such a task is almost never assigned as an exercise for independent work. Nevertheless, such tasks (along with the ones enumerated above) are especially useful precisely as independent work. Indeed, after the teacher has analyzed the conditions and solution of the first problem, its transformation and the solution of the new problem make the pupils consider once more the same relationships and the same quantities, but from another point of view. This facilitates the children's deeper realization of the relationships between the quantities, as well as the methods for solving the problems under consideration.

To minimize the time pupils spend on writing the text of the new problem resulting from the transformation, it is useful, in this case, to use a short outline of the conditions. For example, if they were solving the problem:

> A housewife bought four kg. of potatoes at 10 kopecks per kg. and two kg. of cabbage at 15 kopecks per kg. How much money did she pay in all?

The conditions of the problem may be outlined on the board as follows:

$$\left.\begin{array}{l}\text{Four potatoes at 10 kop.}\\\text{Two cabbage at 15 kop.}\end{array}\right\}\quad ?$$

After the problem is solved, the teacher may underline one of the given quantities--for example, the first--and in place of the question mark, write the unknown which has just been found--70 kop. The children are asked to solve this new problem, and then to transform it, so that it is necessary to compute, for instance, the price of cabbage.

A task dealing with a change in the conditions of a problem which has just been solved is also very interesting and useful. If one of the elements of the conditions of a problem which has just been solved is changed, the new problem may be formulated in whose solution, naturally, this change is reflected. For example, in the fourth grade, children solve the following problem:

> An automobile first went 125 km. and then 1/5 of this distance. The distance covered made up 1/3 of what was left. How many hours did it take the automobile to go the whole way, if its average speed was 50 km. an hour?

In the outline put on the board by the teacher or one of the pupils, one of the elements of the conditions may be replaced--instead of the words "of what was left" the teacher may write "of the whole way" and ask the children to solve this new problem independently.

It is evident from the examples cited that the transformation of the problem itself can be executed by the pupils as well as the teacher and that such a task can have diverse meanings and purposes. The first instance was directed toward the children's better mastery of the interdependence between prices, quantity, and cost. In the second instance, the change was directed toward focusing the children's attention on the meaning of individual words and expressions in the context of the whole problem and on demonstrating how a change which appears small can lead to a large change in the process of solution. It is especially important to make use of this device in the solution of problems which often confuse the children--for example in problems including the expressions "so much bigger" or "so many times bigger" and also when it is necessary to emphasize one or another element of a condition which is essential in the process of the problem's solution. In the second grade textbook [14:23-24], a series of problems in two operations on the increasing and decreasing of numbers by several units is given. In all these problems three quantities are considered. The first is given in the problem, the second is given by how much it is greater (or less) than the first, and of the third by how much greater (or less) it is than the second. The problems require the pupils to figure out the third quantity (or both the second and the third). In all cases (no. 186 and beyond) the comparison is made in exactly the same sequence (the second with the first, the third with the second). Thus the children may stop noticing with which quantity the comparison is made. To emphasize this important point it is useful, after the solution of a problem of the type referred to above (for example, no. 190 [14]: "Draw three columns, one six squares high, the second three squares higher than the first and the third four squares higher than the second. What must the height of each column equal?"), to change an appropriate part of the conditions ("and the third four squares higher than the first").

Such exercises can be used from the first grade, in which tasks in changing of the problem's question also prove to be extremely useful. Here several variations are also possible. The teacher can ask the children to change the question however they like, require for instance, that the question be changed so that the problem is solved in two operations or one operation, or require that the word "larger" appear in the question, etc.

Models for such assignments can be found in the first and second grade textbooks. These exercises may be used for independent work. As a rule (in this case) the children should write solutions to both problems in their notebooks, so that the difference in their solutions is very obvious. It is necessary to check this work in class paying particular attention to the correct formulation of the question.

In the third-and fourth-grade textbooks there are a whole series of assignments dealing with the completion of problems, [15, 16]. problem may be given in which the numerical data is omitted, and the children must fill in these blanks; or the conditions of the problem are given in full, and the children must pose the question and solve the problem.

Exercises in formulating a question for given conditions are indispensable as preparation for the solution of compound problems. These exercises develop in the children appropriate skills which, in many ways, determine the success of their solution of these problems. However, it is no less important that the children be able to select the data necessary to answer one or another question. This ability is important not only for the solution of academic problems, but also as preparation for the solution of problems in real life.

Actually an arithmetic problem has bearing on life whenever it is necessary to answer some question which can be answered by using one's knowledge in arithmetic. First the question arises, and then the data necessary for its solution are selected. To facilitate the children's work, one can point out to them the data from which they will have to select the necessary data and construct a problem. For example, an assignment is formulated as follows:

> Construct a problem in which it is necessary to figure out how much more one housewife paid for her purchase than another one did. The numbers can be selected from the ones written on the board (or on a sheet of paper) or the data in the tables (price lists, tables of speed, etc.).

Much extra time is required to write down the text of the problems that have been constructed. But the teacher, nonetheless, must check on how each pupil handled the task. For this reason, the construction of problems must be combined with their written solution. In such a case, it is important that the questions to each operation, or short explanation of them, be written down. Then, from what is written in the notebooks, it will be easy for the teacher to determine the kind of problems each pupil constructed.

Pupils' independent construction of problems is not only one of the devices for instructing children in problem solving, but also the most important means of strengthening the connection between arithmetic instruction and life, enriching the pupils' life experience, broadening their horizons, and preparing them for the solution of various practical problems. Of late, this sort of task has been given much attention in the methodological literature. The appropriate assignments are well known to every teacher and are enumerated in almost every methodological handbook. They are the construction of problems by analogy, from a given solution, from an outline of the conditions, from given numbers of a definite kind (for example, on division according to content), on a definite theme, etc. However, not all of these exercises are equally suited to the organization of independent work by children. Let us consider, for example, the construction of a problem from an outline of its conditions. Here, the pupil must write down in full the text of the problem, and the teacher must then check 40 of these texts. For this reason, it is best to conduct this task orally as collective work under the teacher's guidance.

Teaching children independent construction of problems from a sketch, diagram or drawing has great educational and instructive significance. Moreover, when children acquire this skill, the teacher can make use of a large number of additional problems (taken from other collections, constructed by the pupils themselves or by the teacher). The fact that each pupil does not have the full text will cease to be an insurmountable obstacle to conducting the appropriate independent work.

Here we have considered, of course, only the basic, typical kinds of pupils' independent work on problems, which allow the most diverse variations.

Independent Tasks of a Practical Nature

As was mentioned above, this sort of work is very rarely conducted in the elementary grades. Nevertheless, the curriculum for each grade acquaints the children with an ever-broadening circle of the units of measurement and equips them with the ability to use units of measurement practically in measuring.

Moreover, according to the arithmetic curriculum for each grade, the children must acquire the elementary skills in draftsmanship. Thus in the first grade, the children must learn not only "to measure a given line segment in meters and centimeters applying the expressions "equal," "greater," or "smaller," "approximately," but also "to draw by eye a line segment 1 m., 1 cm., long; to determine with exactitude, by eye, distances up to 1 m. in class," and so on. In the upper grades the children must learn to use a ruler and set square for drawing angles and the simpler geometric figures (a square, rectangle, and triangle), and to learn to compare line segments by subtraction and division, etc.

The formation of all these abilities and skills can be accomplished only through a sufficient amount of independent practice. It is impossible to learn to draw and measure simply by observing how others do so. These skills are developed in the process of independent work. Observation shows that practical tasks are usually conducted only in lessons devoted to the consideration of new units of measurement. Thus, in the first grade, let us say, the teacher may allot two lessons to acquainting the children with the centimeter and two lessons to the meter. In the course of the two days, the children are occupied with independent measuring (of line segments, various distances in the classroom and at home), but after this they are not given assignments of this kind and encounter new units of measurement only during the solution of problems in the text. Approximately the same thing happens in the second through fourth grades. Here the children use the ruler and set square, as a rule, only in lessons especially devoted to the study of geometric material of new units of measurement. As a result children going into the fifth grade have a very low level of measuring and drafting skills. Checking the work disclosed such scandalous facts as several fourth-grade pupils measuring the length of a line segment with a ruler started the reading not from zero, but from one!

The practical trend in the teaching of arithmetic, dictated by the problems facing our school, requires serious changes in methods of teaching children measurement and draftsmanship. The ruler, set square, and pencil must become the pupils' constant aids in all lessons in arithmetic and shop. Thus, beginning with the first grade, it is necessary constantly to organize exercises which demand the ability to draw a line segment of a given length and then squares and rectangles of given dimensions or, on the other hand, to measure the length and width of a rectangle, and the like. All this work must be closely connected with the study of the arithmetical material.

In the first grade, it is useful to introduce the children to the centimeter significantly earlier than recommended in the textbook. The centimeter is much easier to use in first-grader's independent work than the meter. We would visually acquaint the children with the centimeter in the very beginning of the year—for example, in connection with the study of the number two (two squares in a notebook give approximate representation of a centimeter). It is possible, then, to use this unit literally in every lesson with the introduction of a strip of the given dimension, illustrating one or another number or component of a number. Thus, if the teacher tells the pupils, for example, to "draw a strip four squares long," after having introduced the children to the centimeter, he can formulate the same assignment by telling the pupils to "draw a strip 2 cm. long," etc.

In lessons in shop, instead of operating with the square as the unit of measurement, he can ask the children to measure the length and width of a sheet of paper from which they are to make a bookmark with the help of a centimeter ruler.

Familiarity with the centimeter opens up considerably greater possibilities in this respect than familiarity with the meter, but the meter, too, must be used to introduce as many independent exercises in measuring as possible. In the second grade, when explaining the concepts of comparison by subtraction and by division by the solution of problems in increasing and decreasing a number by several units, teachers widely use graphic illustration of the conditions of the problem. The children's independent execution of all the sketches during the solution of the problems can be handled so that they also become exercises in draftsmanship and measurement. But in the classroom and at home, it is useful, for example, to ask the children to solve several problems of this type:

> One ribbon is 15 cm. long, the second 3 cm. longer than the first, and the third 3 cm. longer than the second. What is the length of the third ribbon?

Instead of writing down the solution, the children can make paper ribbons according to the conditions of the problem, and merely write the numbers 1, 2, and 3 on them. In this case, when correcting homework in class the children should be asked to measure the length of the third ribbon. The same task can be executed in notebooks (the children draw the appropriate line segments).

It is written in the second grade curriculum that the pupils must be able to "construct in their notebooks, by the squares, a square and a rectangle with given sides." Such exercises can also be conducted in connection with the solution of arithmetic problems, and not only as an independent exercise. For example, it is possible to give the children an assignment like:

> Construct in your notebooks a rectangle, according to the squares, with a width of 3 cm. and a length of 2 cm. more than the width, and write down the length of this rectangle.

In the third grade, both drafting and measurement can be used analogously in the solution of problems. They also have a place in work on the topic "Familiarity with Fractions," in which the children can illustrate a half, a quarter, and an eighth, not only with line segments, but also by dividing a square or rectangle into parts. On a practical level, such tasks must certainly be carried out in shop lessons in making various articles, as stipulated in the curriculum for each grade. In the third grade, the children must learn to construct a square and a rectangle using a ruler and a set square. For this reason, it is important to use unlined paper both in arithmetic and in shop lessons, so that the exercises analogous to those described above are really executed by means of these instruments.

Apart from the graphic illustration of the conditions of various arithmetic problems, other special exercises are also given. These require the calculation of the sum of the measures of the sides of a rectangle and a square. The solution of these problems, as a rule, is better introduced not on an abstract level, but connected with drafting and measuring. Rectangular sheets of paper, which the children have previously prepared under the teacher's guidance, can be widely used for the children's independent work. These sheets are given to the children with the assignment to measure the sides, determine what the figure is (square or rectangle), and calculate the sum of the measures. In one lesson, the children individually draw rectangles and squares of given dimensions; in the next, they measure the sides of these figures. With this approach during the consideration of any arithmetical question or the execution of any assignment in a shop lesson (and even in other lessons), the children are compelled to use drafting instruments constantly and conditions are created for the formation of the appropriate skills.

In arithmetic instruction in the elementary grades, every occasion for broadening the pupils' horizons, for familiarizing them with various sides of life (industry, agriculture) should be used. This has great educational significance. Above we have already mentioned that, during instruction in problem solving, this purpose is met through assignments which require the use of diverse reference material, excursions, etc.

Assignments whose special purpose is to impart information to the children, and to form the abilities and skills which are useful in everyday life and in work are also useful. Examples of such assignments can be found in the textbook by A. S. Pchelko and G. B. Polyak for each grade [13, 14, 15, 16]. Here, the making of simple estimates, computations, accounts, the ability to draw a simple diagram, etc., are of importance. Analogous assignments can be successfully used for the children's independent work. As always, the development of the appropriate ability requires the systematic repetition of exercises. In the textbook there are not many of these assignments. For this reason, they must be enriched by the wide use of local factual material.

Calculations made in everyday life of each family can become the objects of consideration in arithmetic lessons. The children must arrange to collect appropriate factual material and the teacher must arrange to help them to select and work on this material so that it can serve as a basis for the independent solution of problems-- for independent calculations. Exercises introduced with this purpose can also be used for developing in the children several skills for work with a book. The teacher can present the task so the arithmetic textbook becomes for the pupils not only a collection of arithmetic exercises, but also a source of various pieces of knowledge about life. This purpose is met through assignments which require the children's independent search in the textbook for the numerical material which characterizes the various sides of life. The teacher can, for example, ask third-grade pupils to look over page 83 of the textbook, select from the text of the problems data concerning the weight of a bag of flour and various grains, and to construct their problems using this data.

In other instances, the children can be asked to select from the text of the problems numerical data which permits them to relate the standard quantity for feeding animals, the standard quantity for sowing various seeds, the freight-carrying capacity of various types of transport, the labor productivity of workers of various professions, etc.

Finally, a place in arithmetic lessons must be found for questions which people often come across in everyday life, but which can not be solved by purely arithmetic calculations--they may be called conditionally, problems on understanding. For example:

> A housewife bought 3 liters of milk for herself and 1 liter for a neighbor; she brought the milk in one can. How can they take out the neighbor's one liter of milk if they do not have a liter measuring cup?

Let the children propose various solutions with the help of any small dishes--cups, mugs, etc.--divide all the milk in half and then in half again, or measure off one liter with glasses, etc. The teacher can, in the course of the discussion, introduce new conditions which make the task more complex. Such practical questions excite great interest in the children, and moreover, are very useful not only in the practical sense, but also for developing powers of observation and understanding in children.

A System for Conducting Children's Independent Work in Arithmetic Lessons

General Questions

In the preceding chapters the forms, the times, the purpose of pupils' independent work in various stages of arithmetic lessons and in all stages of the study of new educational material were clarified. Many examples of assignments for children's independent work, directed toward the formation of various skills, habits and knowledge, were cited. However, to increase the effectiveness of the instruction, it is extremely important that such tasks be organized according to a definite system. To define this system we must consider the various aspects of the work.

The selection of exercises for pupils' independent work, the establishment of their sequence, and the method of their execution is determined primarily by the goals which are being pursued in the study of each topic in the arithmetic curriculum and by the problems of each concrete lesson. For this reason, in the development of a system of independent tasks, we first proceeded in general, from the analysis of the goals and problems of instruction in elementary arithmetic and then, in particular, the problems for each topic and each lesson devoted to the study of these topics. However, not only each lesson, but also the system of lessons devoted to the study of individual topics were considered not in isolation, but as one of the links of a single process of arithmetic instruction in the elementary grades.

The system of independent tasks must provide the children with basic general methods for approaching the solution of any mental problem. These methods are directed toward the recognition and analysis of the problem conditions, and helps disclose the connection between what is known and what is unknown which permits the projection of the course of solution and correctly and exactly brings it to a completion. To ensure the employment of these basic general methods in the solution of problems, assignments for independent work must be given during instruction in a sequence by which the children gradually learn each separate method as well as the ability to apply them in combination. For this reason, the gradual complication of the assignment may move toward mastering various individual methods, as well as toward increasing the requirements for their combined application. Here, it is important that the selection of problems and the nature of the assignments for children's independent work provide variety in form and content to exclude the possibility of the establishment of a set formula for their solution.

Among the most important features in developing a system of tasks for children's independent work and methods for their execution are the characteristics of the children's age and the level of their preparation. It is perfectly clear that pupils' independent work, even when it is based on the same arithmetical material, will be conducted completely differently, let us say, in the first grade than it is in the fourth grade. As an example, it is sufficient to cite the discussion of the commutative property of a sum, which the children first encounter in the first grade and to which they must return in the third and fourth grades. Independent work directed toward the realization of this law of addition can be successfully conducted in the first grade. But independent work is also necessary in the following stages of instruction. The content of the assignments for the independent work and the methods for conducting it in class will, without a doubt, differ greatly depending on the age and previous preparation of the pupils.

While thinking through the content and sequence of the exercises for children's independent work, one must constantly remember the principle for the selection of the exercises and the organization and methodology of conducting them, so that a gradual but systematic increase in difficulty of the assignments is provided. Only under these conditions will independent work foster the children's development in the process of instruction. Increasing the difficulty of the assignments must not, however, lead to the destruction of the requirement that they be within the children's capabilities.

Finally, in determining the content, nature, and methodology for conducting children's independent work, it is important to take into account the characteristics of various kinds of independent tasks. In the first chapter, the types of independent work were named which can be classified according to the pedagogical goals for which they are intended, according to the nature of the educational materials, and according to the activity which they require from the pupils. In a system of independent tasks, all such types must find a place in the process of instruction.

It seems to us that the considerations enumerated do not require additional explanations. Their practical application is given below during the examination of the system for pupils' independent tasks in arithmetic lessons in the first and second grades. However, it would seem useful to illuminate in more detail the various trends in the methods for gradually increasing the difficulty of assignments for pupils' independent work.

Let us consider the most important of these trends. First, in order to execute any assignment whatever, it is necessary that the child possess elementary habits of independent work—the ability to organize his place for work, to select the necessary materials, to use these materials, to understand the teacher's assignment, and act accordingly.

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These abilities are acquired by children in the process of work carried out, first, under the direct guidance of the teacher—who dictates literally every step to the children--and later with an even greater degree of independence. The increase of these demands must appear in the added complexity of the assignment itself, as well as in the additional complexity in the organization of the children's independent work in class. This increase is demonstrated with specific examples. Let us begin with the most elementary--work with a book and the execution of work in a notebook. At first the children are not able to get their bearings in a book (even on a particular page), or in a notebook. The teacher must take care that in each pupil's book the necessary page is marked with a bookmark, and check whether the children have understood precisely which picture on the page must be examined at a particular moment. When giving an assignment for work in notebooks, it is also necessary to specify every trifle--how many squares should be left from the top edge of the page or from the preceding writing, how many squares must be skipped between drawings or written examples etc.

In lessons in the first grade, however, a great deal of time is spent on just these instructions which deal with the technical side of the work and its external form--this is easy to understand. But if such tutelage is not gradually eliminated, the situation will be artificially extended for too long a time. Up to the fourth grade, and even the upper grades, pupils often ask questions about how to arrange the examples, whether they should write the date and how it should be written, whether it is necessary to write the answer in the problem's solution, etc. Such questions bear witness to the fact that the teacher did not make systematic and gradually increasing demands on the children in these matters.

Let us see how increasing the complexity of the assignments for children's independent work on various educational materials is manifested. Consider the illustration of the conditions of a simple problem. In the first steps in the first grade, the appropriate work may, for example, look as follows. The example problem is:

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> On one plate there were five apples and on another three apples. How many apples were there on the two plates?

After reviewing the conditions, the teacher may ask the children to open their arithmetic cases, and remove as many red circles as there are apples on the first plate and to put them in one row. When this assignment is executed, the teacher checks to ascertain whether everyone has done it correctly. He then asks the children the following questions: "How many red circles did you put on the desk? Why did you put out five circles?" and asks the children to count their circles again and check to see if each child really put out five circles. Then, reminding the children of how many apples were on the second plate, the teacher gives the assignment to take as many circles as there are apples on the second plate from the case, put then in a second row, and so on step by step. This is repeated until the children thoroughly grasp the meaning of an assignment and become proficient in the appropriate skills for working with didactic material--until they learn to execute every such assignment quickly and accurately.

After the procedures immediately above are grasped, analogous work will proceed differently. It would be enough for the teacher to say, for example "show with the sticks how many hammers were bought first and how many later" during the solution of problem no. 54--"For the school they bought two hammers and then so many more. How many hammers were bought in all?" Later the assignment will be formulated in still less detail.

In second grade, after the graphic illustration of several problems on increasing and decreasing numbers by several units, and also during the solution of the next problems of this kind, the children can be asked to make a sketch of the problem's conditions as an assignment for independent work. This gradual "curtailment" of instructions from the teacher represents an increase in the demands made upon the children's independent work. This is the case with every other type of assignment.

It is now demonstrated what is meant by the gradual increase in the complexity of work in terms of its organization in a lesson. For the first two or three weeks in the first grade, it does not work well to organize children's independent work on various individual assignments. Because children entering school have not formed habits for such work, it first must be carried out with one and the same assignment for all the children. This is the simplest form of organization under which it is easy for the teacher to interrupt the children's independent work and give them the necessary assistance, give supplementary explanations, etc.

Gradually, as individual children acquire the necessary skills and habits of work, it is possible to individualize the approach so that the majority of children work on one assignment with one set of material, and the others on individual assignments. Ultimately, all the children learn to work with individual assignments. In the upper grades, more and more elements of self-guidance, and even mutual guidance, in the course of the work are introduced into the children's independent work.

The second way in which the complexity of assignments for children's independent work can be increased is dictated by the natural increase of the complexity of the educational material with which the work deals. Thus, for example, although at the beginning of the first grade, the children solve only the simplest problems on finding sums and remainders and then on increasing and decreasing numbers by several units, beginning the second term the children are given compound problems for independent work containing problems of the type considered earlier. Although in the first grade the children solve only problems in which a number is increased or decreased by several units only once, in the second grade, compound problems in which a number is increased or decreased twice are introduced. If, in the second grade, the children deal chiefly with simple problems which can be solved using one operation and compound ones in which it is relatively easy to determine the course of the operations, in the next grades they must become familiar with standard problems and also with compound ones which contain simple problems of diverse kinds and types. This same thing is true, on the whole, with respect to the solution of examples and all other kinds of work whose increase in complexity is dictated by the curriculum itself.

This gradual increase in the difficulty of the educational material with which the children must deal in the course of independent work creates the conditions necessary for strengthening and perfecting the knowledge, skills, and habits which were formed in the previous stages of instruction. However, it would be wrong to suppose that this trend in the organization of children's independent work is predetermined by the curriculum and that the teacher need not think about it specifically--that it will come about all by itself, with progress through the curriculum and the textbook. The fact is that, despite all the attempts of the textbook authors to construct books so that they provide for the systematic consolidation and review of what has been covered, in very many cases of the progression to the consideration of new topics, the educational material not only does not grow more complex, but, on the contrary, becomes simpler. This is demonstrated with concrete examples. Working from the first-grade textbook, the children, after working for a long time with the solution of compound problems in addition and subtraction, turn to the study of multiplication and division. The curriculum does not provide for the solution, in the first grade, of compound problems which contain these operations. For this reason, the appropriate sections contain only simple problems. The sections in the textbook which deal with multiplication and division do not provide sufficient material for the consolidation of the knowledge and skills which were formed in the children during instruction in the solution of compound problems (individual problems of this sort are given here only in the small "Review" sections). If the teacher loses sight of these considerations and does not provide for the systematic continuation of work on compound problems, the knowledge, skills, and habits related to their solution and previously acquired by the children not only will not be perfected, but may be lost to some degree.

We take our second example from the fourth grade. One of the most crucial general topics in the fourth year of instruction is "The Four Arithmetical Operations on Numbers within the Bounds of a Million" (in the curriculum called "Whole Numbers"). The study of this topic occupies the whole third quarter of the school year. The subtopics numeration, addition, subtraction, multiplication (by a one- and two-digit number, and then by a three-digit number), division, and the order of the operations are considered consecutively within the more general topic. The problems and exercises in the sections of the textbooks devoted to each of these topics are selected in strict accordance with the topics. Their selection is determined by the problems of explaining the meaning of each operation, its applications, its properties, and its relations to other operations. However, if the children are limited to the solution of these problems and examples, it is possible to hinder the consolidation and, above all, the further development of the children's habits of independent work on more complex material, with which they dealt previously (specifically, the more complex standard problems).

Finally, it is essential to provide a gradual increase in the difficulty of the assignments directed toward the formation of various skills. Tasks which are diverse in nature and which demand some degree of independence from the children can be carried out on the same arithmetical material and with the same lesson organization, depending on the way the assignment itself is formulated. It is one thing, for example, if the teacher, having chosen one or another problem for the children to solve independently, tells them directly that, before solving the problem, they should make a sketch of its conditions; and it is another thing if he lets them settle the question of the most suitable form for such an illustration independently. It is also quite different if, having asked the children to construct all possible examples for a given answer, the teacher reminds them that they should use all four arithmetical operations instead of not saying so at all.

The increase in the complexity of the exercises must be linked with the fact that in order to develop the individual skills and habits from which general ability is formed--to solve a problem independently--it is necessary, in the end, to bring the children to use these particular skills in combination. For this reason, independent tasks can at first be composed of small assignments, directed toward the development of the ability to read conditions independently, to illustrate them, to select the data needed to answer the question, etc. However, later they must be gradually replaced by assignments which require application of various methods and means of analyzing the conditions and searching for the way of solution, which the children acquired in the course of previous exercises.

Thus, the gradual increase of the demands made on the pupils in the system of independent work conducted in arithmetic lessons can be linked with the increased complexity of the form in which these tasks are organized, with the content and form of the assignments, with the change in the nature of the work in terms of the activity which the tasks demand from the children, and also with the methods of conducting the work.

After these general remarks, we will now consider concrete methods of implementing these requirements in the system of pupils' independent work, in arithmetic instruction, in the first and second grades.

Since the basic factors determining the system of these tasks, as we mentioned above, are the goals and problems in the study of each topic of the curriculum, and the pupils' level of preparation, we will consider the content, organization and methods of conducting children's independent work on material from the basic topics of the curriculum. Within each topic, we will single out questions dealing with the system of instruction in problem solving, with work directed toward the children's mastery of arithmetical operations, and with exercises whose goal is the formation of the basic arithmetical concepts.

Pupils' Independent Work in the First Grade

Children's independent work in lessons on the topic "The First Ten Numbers." The topic "The First Ten Numbers" is divided into three subtopics, each of which is characterized by its own specific problems which determine, to a significant degree, the content and nature of the children's independent work in each of these stages of instruction. For this reason, we will consider each of them separately.

The preparatory method pursues the following goals: to ascertain the preparation with which each pupil comes to class, to systematize, enrich and deepen the knowledge of numbers and computation which he acquired in the preschool period, to lay a foundation for the formation of the appropriate generalizations at the next stage in the study of numbers and operations within the bounds of 10, and to familiarize the children with the elementary laws of academic work in class that is, to prepare them for regular, systematic instruction. Literally, from the first lesson, the teacher faces the task of teaching the children how to learn. In the introductory lessons it is necessary to teach the children at least to consciously apprehend and exactly execute the teacher's simple assignments dealing with the use of a book, notebook, demonstrational visual aids, and didactic material distributed to each pupil. The content of the children's independent work in these lessons will primarily consist of counting objects and practical exercise with didactic material directed toward the formation of basic arithmetical concepts--the concepts of "as much as," "more," "fewer" (in an arithmetical sense).

Since one of the main goals of the introductory lessons is to ascertain the knowledge, skills, and habits which the children possess, the children's independent work in these lessons may have not only an instructive, but also a verificational nature. As a rule, lessons considering each new question should begin precisely with verificational independent work, since from the results the teacher will be able to correctly estimate the knowledge the children possess about this question, and construct the next lessons.

However, at the next stage the teacher faces the task of filling in the gaps disclosed in the children's knowledge. For this reason he naturally turns to systematic instruction. In explanation, the teacher uses the appropriate visual aids, and the children observe his actions. Most frequently, pupils' independent work in these lessons consists of the most exact reproduction of the teacher's actions of which they are capable. In the present case, the main demand made on the children consists of following instructions as closely as possible, and of obtaining results close to the model the teacher set forth. In both their content and their character; assignments at this stage of instruction are of the most elementary nature. They are very short--a task with didactic materials lasts, as a rule, for two or three minutes. However, in a lesson several such tasks (four or five by our observations) may be carried out. Tasks dealing with drawing in notebooks may last for a longer period, since they are training exercises and consist of the repetition of the same element.

The gradual increase in the complexity of the assignments may begin in these lessons. As illustration, one of the first assignments is:

> Accurately, using a ruler, line up six circles on the desk.

The following, more complex assignment deals with the formation of the concept "as many as":

> Line up in a row, using a ruler, five circles and below them in a second row as many squares.

In executing the latter assignment, the children use the method of making up equal groups, with which they have been acquainted by the teacher (the alignment of the elements of the two groups in one to one correspondence; each square must be placed below a corresponding circle). What is being demanded of them now is not the simple reproduction of the teacher's actions, but the application of a method they have learned under different circumstances (The teacher may demonstrate this method with demonstration materials, using, for example, various pictures of objects, and the pupils, in their independent work, use circles and squares. The teacher may conduct the demonstration with groups consisting of six or seven objects, but the children are asked to do the same thing with groups of five objects, etc.). The following, still more complex assignment is:

> Take several circles and several squares—as many as you like. Find out if there are as many circles as squares.

The difficulty of the exercise is determined by the fact that, in carrying it out, the child must "guess" that to answer this question he can use the same method of aligning elements of both groups in one-to-one correspondence. Experience shows that, under these conditions, children frequently return to the use of more familiar methods of comparison based on the estimation of the given groups "by eye"--by spatial indications (Some children, for example, pile all the squares one on top of another and make another column of all the circles and judge whether there are as many squares as circles from the heights of these columns. Others lay the figures out in parallel rows, without observing the principle of aligning them piece by piece, and thus they often come to an incorrect conclusion based on whether one row is longer). In this way, we see that such an assignment compels children to choose a way of solving the problem, thus developing the skill of applying acquired knowledge to new conditions.

The selection of assignments for the first arithmetic lessons must also fulfill the requirement that the children's independent work be varied. This requirement is determined, on one hand, by the psychological characteristics of children entering school (extremely unstable attention, interest easily excited by one matter or another and just as easily lost), and on the other hand, by the task of creating a sufficient sensory basis for the formation of appropriate arithmetical concepts. It is well known that one of the most important conditions for the formation of these generalizations is the accumulation of a rich experience with practical operations, using diverse groups of objects. From this it follows that independent work in these lessons must be constructed using diverse visual materials.

Since we want the pupils to formulate several generalized laws of operation (here we have in mind the method for comparing groups of objects described earlier), even at this stage of instruction it is necessary to vary more than just the objects. All conditions which are not essential from the standpoint of the particular law must be varied. A simple reciprocal correspondence between the elements of the two groups being compared may be established in various ways. In one case, for example, the circles are placed on top of the squares (each circle on one of the squares); in another, the squares and circles are laid out in two parallel rows, one below the other; in a third, in two vertical rows. Finally the alignment is conducted so that the pupil takes from a box containing both circles and squares, a pair (a circle and a square) at a time and puts them into another box, until it becomes clear whether there are any extra circles or squares.

All these exercises will facilitate the children's realization of the principle at the basis of this method of comparison; while, if the appropriate exercises are conducted in a monotonous form, the essential feature may easily be replaced by a non-essential one (as when, in the example cited above, the pupils lay out the objects of both groups in parallel rows, without observing the principle of a one-to-one correspondence of the elements).

The teacher should guide the children in their familiarization with all the ways of arranging the objects of the groups being compared. However, this does not mean that the teacher himself must, in all cases, initially demonstrate each of them. Depending on the pupils' preparation, it is sometimes possible for the teacher to limit himself to instructions: "Compare and find out which there are more of, squares or circles, but do not lay them out in two rows. Put the circles on top of the squares." It is even possible, in some cases, to ask the children to figure out by themselves how else it would be possible to find out whether there are as many circles as squares. Under these conditions, from the very beginning, children's independent work in arithmetic lessons demands from them not only concentrated attention but also independent thought, and makes it possible for the children to manifest their own initiative.

We have described the features of children's independent work in the introductory lessons from the point of view of their arithmetical content. However, one must also consider their organizational features. One of the main goals of the preparatory period, as we noted above, is the formation in the children of elementary habits of academic work. In these lessons, on the basis of independent activity, the children must learn the fundamental methods for working with the material distributed to each pupil and with the sticks (how to keep them, get them out, put them on the desk, etc.), and learn to use a book (bookmark, pointer, turning pages, finding the necessary material on a page, etc.). One must familiarize the children with lined (squared) notebooks, teach them to outline the squares and independently execute other uncomplicated assignments dealing with the use of lined paper, and prepare them to write the numerals (the drawing of simple "borders," the writing of the elements of the numerals, etc.).

While they execute the appropriate independent work, the children, as a rule, act from the model the teacher has set up and in accordance with his instructions. The increase in the requirements must be accomplished here by a shortening, a "curtailment," of these instructions which were mentioned earlier.

In terms of organization, the independent work in the introductory classes, as a rule, is general--all the children work simultaneously on executing the same assignment from the teacher. We became convinced in practical situations that it is not worthwhile to strive for individual assignments at this stage of instruction, although, at first glance, (taking into account the difference in the preparation of children entering first grade), this may also seem alluring. We learned that, in practice, when children's individual work is started too early, the formation of a most important academic ability—the ability to listen to the teacher when he addresses the whole class—is hampered. As M. N. Volokitina very correctly noted in her study, one of the difficulties of the first lessons lies in the fact that "some newcomers, at times, do not realize their roles in class and look on all the proceedings as spectators, without relating the teacher's words and demands to themselves" [22:7-8]. The teacher, in giving individual assignments, may often foster a situation in which many of the children constantly demand special attention and become used to working outside the group. This is one of the features of the preparatory period of instruction relevant to arithmetic lessons.

The next stage of instruction deals with the study of the first ten numbers. The fundamental educational goal in this stage of instruction is the formation in the children of a clear idea of the first ten numbers—their formation, composition and relative magnitudes. The children must practically master the concept of number and the axiom of counting (the results of counting do not depend on the order in which the objects in a given group are counted), and become acquainted with the series of numbers and learn to compare numbers by the place they occupy in this series. Finally, they must become acquainted with the numerals which designate the numbers, and learn to recognize and distinguish the numerals and write them. The children must ascertain the relation between quantity, number, and numeral in various combinations. In the study of this topic, it is of great value to familiarize the children with arithmetical problems and examples—their construction, solution, and the writing of the solution.

The system and methodology of the study of the first ten numbers are clearly reflected in the textbook. Our task is not to consider its substance. In building a system of children's independent work at this stage of instruction, we took into account the content and methods of work which are determined by the curriculum and textbook. The numbers are considered one after another; in the consideration of each new number analogous questions are analyzed. Because of this, conditions are created in the instruction process under which similar assignments are repeated in lessons devoted to the study of various numbers. This permits the gradual increase in the demands made on the pupils and helps them to manifest more and more independence in the consideration of analogous questions and in the development of their activity. Lessons on the study of the first ten numbers must be used maximally with a view toward further perfecting the children's ability to work independently with pictures, books, distributed material and notebooks. The task of forming habits of independent work is one of the essential goals of the period under consideration.

Let us now consider the possible conclusions, concerning the nature of the material on which the children's independent work is built, which ensue from the goals (formulated above) of this stage of instruction. The fundamental task, as we noted above, is to familiarize the children with number. While one should not strive to develop in the children the concept of number at this stage, one must make them conscious of the central arithmetical fact that the result of counting does not depend on quality, on the individual features of the objects counted. It is, of course, self-evident that we are not talking about the deduction of a law--not about the children's formulation in words of this proposition--but only about their inner conviction of the possibility of replacing some objects with others in solving the problem of their quantity. Thus, if a problem mentions objects which are not at hand, it is possible to use circles or squares in their place in solving the problem. If squares are mentioned, it is not mandatory to count squares--they may be replaced by counting sticks, and so on. It is perfectly clear that the children can arrive at this conclusion only if, in their own practical activity, they have the opportunity to become convinced repeatedly of its correctness. Accordingly, the necessity for organizing the children's independent work with diverse didactic materials so that it will lay a foundation for the appropriate generalization becomes clear.

As an example of tasks of this sort, we may cite the execution of such assignments as:

> Put 5 squares in a row, below them put as many circles, below them as many triangles, and below them as many sticks.

In checking the work, it becomes clear how many circles, triangles and sticks were put out and why (because it was necessary to put out 'as many,' that is five, since there were five squares). As a check of whether the proper generalization was made—of whether the children can transfer the regularity they have noted to other instances not illustrated by visual aids--the teacher may ask a central question: "And if I tell you to take as many notebooks, how many notebooks should you take?" (Notebooks are out of sight at the moment). If the pupils answer "also five" it serves as sufficient proof that the required generalization has been formed in their minds.

Furthermore, independent work with this aim must contain all possible exercises in combining two groups of objects and detaching a part of a given group of objects. The recognition that the number of objects in a group formed by combining two smaller ones does not depend on the objects with which the operations took place, serves as a basis for the formation of concepts of number. Experience with such practical operations using quantities of objects is also indispensable for developing in the children an understanding of arithmetical operations. The corresponding independent tasks for the children will resemble the one described above. In them, the knowledge and abilities which the children have acquired during the first lessons receive subsequent development. During the whole period in which the first ten numbers were studied, the necessity of the children's comparing two groups of objects by the number of objects in each group does not disappear, since it is the basis for comparing numbers. Such tasks, familiar to the children from the preparatory demonstrations, do not require that the teacher give a preliminary demonstration of the appropriate operations. They can serve as a starting point in examining the formation of a new number and in comparing it with the preceding one.

In this stage of instruction the next step forward must be made. The children will receive a more complete idea of each of the first ten numbers and will recognize several traits of numbers on an abstract level--the possibility of expanding a number into the addends which compose it and the traits which follow from the place each occupies in the series of other natural numbers. Here it is always necessary to reckon with the concrete thought processes of younger schoolchildren and their insufficient preparation in operating with abstract material. Throughout the period during which the first ten numbers are studied (and sometimes even later), in the consideration of every such trait, it is necessary to start from practical operations with real objects. The children can be taught the necessary generalization only on the basis of accumulated experience of such operations under suitably varied conditions.

For example, to explain the possibility of expanding a given number into its component addends, it is useful to organize practical work in separating suitable groups of objects into components. The formation of a number may be illustrated by the formation of suitable groups of objects (by adding still another object to a group corresponding to the number previously studied), and so on. All illustrations of this type, with which it is necessary to begin the explanation of any of the questions of the topic under consideration, are, at first, given by the teacher using demonstrational visual aids. However, it is impossible to restrict oneself only to demonstration. The children's independent execution of analogous operations with other groups of objects (using individual didactic materials) is indispensable for the formation of the proper generalizations. The children's independent work is here carried out at the stage of initial consolidation. However, in lessons devoted to each subsequent number, it may be carried out as preparation for the introduction of new material. Sometimes as early as the stages of initial consolidation the children's independent examination of a new instance proves possible.

Above we cited an example of how by increasing the ratio of the children's independent participation to total participation from lesson to lesson, we gradually brought them to realize the principles which permit the enumeration of all the possible combinations of two addends composing any number. It was demonstrated that, with the proper preliminary preparation, the examination of the composition of a new number could successfully be carried out on the basis of the children's independent work with cut-out didactic material, and, later without it, on an abstract level. Thus, as early as the first steps of instruction, with the proper methods and with a strictly gradual increase in the assignments' difficulty, children can independently gain an understanding of questions new to them.

Analogous independent work by children may also serve as a basis for the childrens' familiarization with the formation of new numbers, and for a comparison of the newly formed number with the preceding one. However, it is necessary that, in the lesson devoted to the study of numbers, there be the same gradual transition for the children from listening to the teacher's explanation and reproducing his actions to ever greater active participation in solving analogous problems with the material about new numbers.

In the study of the first ten numbers, along with practical operations with groups of objects, even greater significance is acquired through exercises devoted to the use of numerals. The use of numerals makes it possible to organize the most diverse independent tasks for the children, based on the ability to correlate a quantity and a numeral, and to distinguish and recognize the numerals.

At first, the appropriate exercises are conducted by work with the class as a whole during which the teacher can immediately check the correctness of each pupil's execution of the assignment. Later, the children do the exercises independently. Some examples of such assignments follow:

> Make a square of sticks. Count how many sticks were required, and find the corresponding numeral.

For the execution of independent work on this sort of assignment special cards on which various figures are outlined by sticks may be used. It is also useful to make up individual sheets on which, there are, for example, three circles on one line, five squares on the second, two apples on the third, and a mushroom on the fourth. Such sheets can be used for exercises in distinguishing numbers, and for checking on the children's ability to correlate number and quantity. The children are given an assignment to write the proper numeral on each line. Reverse assignments, in which the children must illustrate with a drawing various numbers represented by numerals, are also possible.

Aside from these assignments, one must devote a great deal of attention and time to work dealing with the formation of the ability to write the numerals. Frequently in practice the writing of numerals crowds out all other kinds of independent work in these lessons. Without a doubt, this does definite damage to both the development of the children's independence at this stage of instruction and the mastery of material in the curriculum. But it is also impossible to neglect the writing of numerals. In the experimental class, exercises in writing numerals were carried out in literally every lesson, but they occupied not more than five minutes.

In our experiment, in the organization of children's independent work, we made wide use of the so-called movable numerals (found in the supplement to the textbook). They were used in examining the composition of a number (after separating a given group into two parts, the children had to designate the number of objects in each of them by means of the "movable" numerals each time), in the relation of number and quantity and in all exercises directed toward the mastery of the numerical series.

A whole series of assignments for independent work is directed toward the study of the numerical series: independently arranging the numbers in order (an exercise which must be carried out systematically); and filling in the blanks in a series of numerals etc. After the children have become acquainted with the signs of operation and the way to write them, they can be given, for independent work, the "printing" of examples from models given in the textbook or on the board, and their solutions. Already, at this stage of instruction, the work can be diversified and not limited to the copying of prepared examples. The children may be asked to compose numerals independently in an assignment as follows:

> Compose two examples in which it is necessary to add one or in which it is necessary to subtract one or, to a given number add a number so the sum is, let us say, three.

After the children have learned to write out the solution of a problem (if only by means of cut-out numerals, without designations), they can be required to compose and solve problems independently. Of course, this task must first be prepared for carefully with appropriate exercises executed under the teacher's guidance. These may look approximately as follows:

> Carefully watch what I am going to do and then compose a problem from what you have seen.

The teacher then, before the children's eyes, puts two trucks and one car on a toy shelf. The children must write the solution of the problem by means of cut-out numerals. By the way the writing is done the teacher can, to some degree, tell whether they have been able to handle the assignment. However, in checking it is necessary not only to look at how the solution is written, but also to listen to the problems composed by two or three of the pupils.

The next step (a more difficult assignment) is the composition of problems from pictures. For this purpose, the pictures in the textbook may be used. Here, however, it is necessary to keep in mind that from one and the same picture at least three problems can be composed.

We will demonstrate this with the example of the picture depicting kittens [1 :16]. From this picture it is possible to compose the following problems.

1. Four kittens are sitting on the floor and one on the table. How many kittens are there in all?

2. In the picture there are five kittens. Of them, three are drinking milk, and the rest are not. How many kittens are there in all?

3. In the picture there are five kittens. Of these, two are reddish, and the rest are gray. How many gray kittens are there in the picture?

4. In the picture there are three gray kittens and two reddish ones. How many kittens are there in the picture?

The assignment must be definite, so that it is easy to check. After asking the pupils to compose a problem from the picture, the teacher can specify:

> Compose a problem from this picture so that to solve it, it is necessary to take away (or add).

In solving problems at this level, when children still have not mastered arithmetical operations as such, but determine the answer on the basis of counting up the sum (or remainder), it is useful to have them make a drawing of the problem's conditions in their notebooks. In the drawing, the children can depict the objects which are being discussed (mushrooms, apples and others).

It is useful, however, to gradually introduce a form of illustrating the conditions in which the objects mentioned in the problem are replaced by others. For example, after introducing the pupils to the problem:

> Yura had eight pigeons. One pigeon flew away. How many pigeons did Yura have left? [13: 25].

The teacher can ask the children to outline in their notebooks as many squares as Yura had pigeons, and cross out as many of them as there were pigeons that flew away (the assignment may be given all at once or in parts, depending on the preparation of the class). After the teacher has checked on how the children have handled this assignment and reviewed with them the conditions and questions, he may ask:

> What sign do you use to write the problem's solution--'to add' or 'to subtract'?

The pupils indicate the proper sign from the selection of numerals and signs which they have. After making sure that the children have chosen the correct operation, the teacher may ask them to write the problem's solution independently by using the movable numerals.

If the teacher has the appropriate materials at his disposal in the study of the first ten numbers it is also possible to successfully organize gamelike independent work for the children. Examples of such games and the materials for them are given in the printed handbook for the two-group school [18].

When the teacher and children are examining addition and subtraction within the bounds of 10, the following must be kept in mind. When the children study the first ten numbers, they become familiar with the division of each of them into addends and can, in many cases, determine the sum of two numbers or their difference based on a visual idea of the composition of numbers. However, they have still not mastered this material to such a degree that they can confidently solve any example on addition or subtraction within the bounds of 100 on an abstract level. In case of difficulty, they can find the solution only by counting the objects (sticks, abacus beads, etc.).

To form in the children the habit of consciously executing abstract arithmetical operations without giving up visual concepts of numbers and without ceasing to use visual aids, it is necessary to put at the children's disposal a method for determining the result of addition or subtraction which permits them to solve any example without recourse to visualization, based only on knowledge of the numerical series and the principles of the operations. At this stage of instruction the goal is to teach the children a method of addition and subtraction by adding on and taking away units and groups.

The sequence of introducing the various cases of addition and subtraction within the bounds of 10 is noted in the textbook. Beginning with adding and taking away 1, 2, and 3, the children gradually come to consider more and more complex cases based on the preceding ones. As a result they should master this method of adding and subtracting and learn the composition of the first ten numbers.

When the children study, one after another, the tables for adding 1, 2, 3, etc., it is important not only to ensure that the children have a firm knowledge of these tables, but also to bring them to an understanding of the general principle on which these means of addition and subtraction are based. Namely, each number may be added or subtracted either by a unit or by groups into which it may be separated. The material of this chapter affords wide possibilities for the formation of this generalization, for the excitation of the children's interest in the composition of numbers, and for the development of the pupils' powers of observation, initiative and quickness of wit. When introducing each new case, the teacher must, above all, remind the children of the composition of the number whose addition or subtraction will be considered in the lesson. He must also review the cases of addition and subtraction of smaller numbers which may be used in adding the given number by its parts. In the lesson various ways of adding and subtracting the given number must be analyzed, but in the end the children, with the teacher's guidance, must explain which of them is the most rational.

We will show how work on the study of a new case of addition may be conducted, using as an example a lesson devoted to adding the number four. While checking homework or mental arithmetic, cases of addition and subtraction which were studied previously must be reviewed. The most attention must be concentrated on adding two and three (in considering examples in which three is added, various ways of doing so should be reviewed: +(1 + 1 + 1); +(1 + 2); +(2 + 1); + 3). In preparing for the new topic it is also necessary to review the composition of the number four and give the children practice in solving examples in addition like those given in the textbook under no. 73: 3 + 2 + 2; 5 + 2 + 2; 4 + 2 + 2; 6 + 1 + 3; 3 + 3 + 1; etc., and to review the series 1, 3, 5, 7 and 2, 4, 6, 8, 10.

The appropriate work may be organized in different ways in the lesson. Thus, the lesson may begin with the children's independent work with an assignment like this:

> Write the numerals for the numbers which compose the number four and then solve, in writing, the examples from the textbook.

In executing the assignment, the children must write:

$$\begin{array}{cccc} & 3 & 2 & 1 \\ 4 & & & \\ & 1 & 2 & 3 \end{array}$$

This representation is familiar to them from the time they studied the first ten numbers. The solution of examples from the textbook may be conducted orally, with only the answers written down, so that too much class time is not spent on an assignment which prepares the children for the consideration of new material. Thus, preparation for the new material is based on the children's independent work.

After the preparation, the teacher may formulate the lesson's goal: "Today you will learn to add the number four."

Two groups of two and four objects are illustrated with demonstrational visual aids. The teacher asks the children, "How can we add four blocks to two blocks?" With the pupils' aid the teacher analyzes various cases in which blocks are added by ones or twos, first one and then three, etc. The same cases are illustrated with other individual and classroom materials. With the teacher's guidance, the children should conclude that it is easiest to add 4 by twos (if you know the table for adding two).

Then the teacher and the children consider the drawings and detailed writing of the solution to the examples 2 + 4 and 4 + 4 given in the textbook on page 40, after which the teacher writes the appropriate representation on the board and asks the children to solve the rest of the examples independently, using various ways of computation. After the children do this assignment and the teacher checks it, problem 75 is solved orally. The children may make a drawing in class for problem 76, which may be assigned for homework. Example 78 may also be given as homework.

In the consideration of adding 5, 6, 7, etc., the device of rearranging the addends may be used. Here too the relationship between the corresponding cases of addition and subtraction must be established practically. This is accomplished by specially selected examples from the textbook. The teacher must take care that the pupils are made conscious of the meaning of the juxtaposition of examples of the type: 4 + 1 and 1 + 4, 4 + 2 and 2 + 4 (from exercise 78), 2 + 4, 6 - 4, 4 + 4, 8 - 4 (from exercise no. 92) and others.

In connection with using the methods of rearranging addends taking into account that the children have already amassed a rather large experience in adding and subtracting--the lesson devoted to adding and subtracting the second five numbers should be constructed so children's independent participation in work on new material increases.

In the lesson whose goal is to introduce the adding of seven, the work may be conducted in this fashion. Preparation for the new material must cover the same questions as before--the composition of the number seven, the review of adding smaller numbers (3, 4 and 5), the solution of examples of the type 5 + 3 + 4, etc--. But it is also important to give several examples in whose solution the rearrangement of addends is used. After this the children may be asked, in the course of their independent work to examine the illustration on page 49 of the textbook, and to solve the corresponding example. They must attempt to draw the conclusion independently but under the teacher's control, who helps to formulate it if necessary. In exactly the same way, the detailed writing of the solution of the examples 2 + 7 and 3 + 7 given in the textbook may be considered first by the pupils themselves, so that they may try to figure out, independently, the method of computation prompted by the written representation. Here, as in the first case, after the children think about this assignment, the teacher listens to them and helps give a clear explanation, which is later consolidated in the solution of other examples. Thus, the gradual increase of the demands made on the children is ensured.

During the study of addition and subtraction within the bounds of 10, the most frequent use is made, in practice, of the written solution of examples as an assignment for children's independent work. However, other assignments, which require a greater strain on their thought processes may also be used with success--assignments involving original composition of examples and the application of acquired knowledge under new conditions.

Thus, in the study of the first ten numbers, the pupils' independent composition of examples from assignments like "Compose four examples where three is added,"[1] and "Compose examples in addition to which the answer is eight," are already within the capacities of the children. The composition of examples from a given answer are especially useful in the study of addition and subtraction within the bounds of 10, since they require a knowledge of the composition of the first ten numbers. This kind of assignment is made simpler if the teacher indicates how many examples should be composed, and made more difficult if the number of examples is not indicated. Such work becomes still more difficult, but also more interesting, if the operation is not indicated and only the answer is given. In this case, for each number within the bounds of 10, nine examples can be composed, if only examples in one operation are considered. If the children have already solved examples in two operations, the number of examples which may be composed from a given answer increases considerably. In this case, the assignment must be given a definite time limit, so that the children are judged according to who makes up the most examples in the given time.

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[1] See [13: problems 34, 46, 67, and others].

The composition of examples from three given numbers is an easy and very useful exercise. For example, the numbers 2, 6, 8 are given. From them the following examples may be constructed: 2 + 6 = 8, 6 + 2 = 8, 8 - 6 = 2, 8 - 2 = 6. Such exercises are the children's first preparation for the discovery of the relation between addition and subtraction. Moreover, they facilitate the mastery of addition and subtraction by tables.

Another good exercise is the solution of examples with one of the addends, the minuend, subtrahend, or operation sign left out (with the answer given). The children are asked to copy examples, filling in the blank.

Aside from those enumerated above, assignments specially directed toward mastering the way to add and subtract by ones and groups are also useful. The textbook constantly gives models for the detailed writing of the solution of examples, models which reveal these ways. Consequently, it is eventually possible to include assignments like the following:

> Solve the following example from the model given in the textbook; In the textbook you are asked to solve the example 8 - 4 thus: 8 - 2 - 2. Solve this problem another way; or Can this example be solved by taking away one at a time? Write this solution.

If the class is sufficiently prepared, and if the pupils easily switch from one type of work to another and quickly master the technical execution of new assignments, it is possible, using the material on the first ten numbers, to acquaint them with circular examples, magic squares, lotto games, and others.

Work which is organized around the solution of examples must also be diverse. Children can work on the solution of the same examples written on the board, solve the same examples but from a book, execute assignments in variants (it is useful to train the children to do this kind of work from the very beginning), or from cards with individual assignments. During instruction in problem solving at lessons devoted to the topic under consideration, work begun earlier undergoes further development. Thus, along with colored cards from which problems were composed, drawings, including elements of diagrams (brackets, arrows) are introduced. The unknown is represented in such a drawing by a question mark. First the children learn to make and "read" such drawings under conditions of group work, and then the composition and solution of problems from such drawing-diagrams is carried out completely independently. The drawings may be of various kinds, a variation upon which depends the difficulty of the assignment. Thus, the drawing may be constructed with the full use of visual aids with objects representing both the given and the unknown data; the children must only count the objects depicted. For example, two dishes are depicted with four apples on one and five on the other. Below the dishes is a bracket, and under it a question mark. There also may be a drawing with less than full use of visual aids, in which the unknown cannot be found by counting (for example, a closed box of colored pencils is depicted and below the box is written "four pencils," and alongside the box three pencils are drawn. Both of these pictures are joined with a bracket, below which is a question mark). Finally, the third type is when in the drawing only single objects of the kind discussed in the problem are drawn, and the numerical data is written (for example, a roll is drawn and under it is written 6 kop., alongside is a tart, and under it is written 3 kop., a bracket and a question mark). The teacher puts such diagrammatic illustrations on the board.

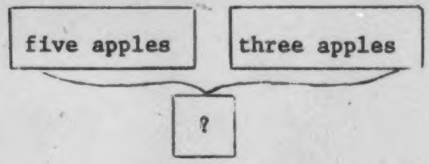
It is still better if the teacher has the appropriate posters. (The children themselves may successfully assist in making them.) Using the posters, more complex work may be carried using the assignment:

> Compose as many problems as possible from the given poster, or Compose two or three problems, and so on.

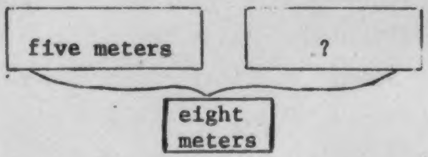
The question of which precise type of illustration is better must be decided according to each separate case, with regard for the children's preparation and the features of the problem.

After the children have mastered the composition and solution of problems from an illustration of the type described, it is possible to replace them gradually with a diagrammatic representation of the conditions (without illustrations). For example, the children may be asked to construct and solve a problem from the diagram:

For Addition



For Subtraction



Work on individual assignments in composing and solving problems is the next step forward with respect to organization as well.

Pupils' independent work in lessons on the topic "The Second Ten Numbers." In studying oral and written numeration within the bounds of 20, the children must acquire a whole series of new facts important from an arithmetical point of view concerning the principle of the construction of the decimal system, the spatial significance of numerals in the notation of numbers, and the decimal composition of the second ten numbers. One of the main goals of the lesson devoted to these considerations is the formation in the children of concepts of ten not only as an aggregate of ten units, but also as a new compound computational unit.

In acquainting the children with the concept "ten" and juxtaposing it with the concept "unit," the teacher should make wide use of the special visual aids constructed so that, by using them, it is easy to emphasize the matter. The special demonstration board divided into pockets--two rows of ten pockets--may be used in demonstrating "ten" as a "unit." Blocks or beads arranged vertically, which may be threaded on two wires (10 on each), and other devices may also be used.

It is important that, aside from this type of demonstrational material, each pupil have analogous materials (best of all are the demonstration boards which permit the use of the diverse cut-out visual material, which is found in the supplement to the textbook). With the help of these materials it is easy to illustrate not only the formation and composition of each of the second ten numbers, but also all the cases of addition and subtraction within the bounds of 20.

In lessons devoted to oral and written numeration, after an explanation of the new material with demonstrational visual aids, the children can independently do a series of exercises directed toward the consolidation of what has gone before. For example:

> Write in your notebooks (or use the movable numerals) how many blocks are on the shelf, how many beads are marked off on the abacus, how many circles are out on the demonstration board etc--or the reverse assignment--Outline in your notebook columns of squares corresponding to the numbers 13, 17 and others, like the model given on page 60 of the textbook—or—from the series of numbers written on the board, select and write with movable numerals only the ones which contain 1 ten and 7 units; write the biggest of the numbers, the smallest of them.

Finally, besides solving examples like 3 + 10, 16 – 10, 17 - 7, exercises in the expansion of numbers into two addends, one of which is 10, may also be included in the assignments for the children's independent work. The assignment may be given in the following form: Write as in the model:

17 = 10 + 7   14 = 10 + 4
13 =   15 =

These exercises will be the best preparation for considering the fundamental methods of addition and subtraction within the bounds of 20.

For each lesson devoted to addition and subtraction within the bounds of 20--in the simplest cases when one of the addends equals 10—the work begins with exercises in the division of numbers into tens and units, and the formation of these numbers from ten and units (14 is 1 ten and 4 units or 14 units, 10 plus 4 makes 14, 1 ten and 6 units are 16, etc.).

In exactly the same way, for instruction in subtraction one should proceed from the partition of numbers into the tens and units which compose them. This device is also used in lessons devoted to addition and subtraction without carrying over tens. As preparation for the consideration of new material, addition within the bounds of 10 and the composition of the second ten numbers are reviewed in these lessons. The appropriate assignment may be given as the pupils' independent work, since they have executed analogous exercises earlier. The teacher should, however, check the correctness and awareness of the children's execution of this assignment, and only then turn to an explanation of the method of addition without carrying over ten within the bounds of 20.

In the explanations not only class, but also individual, visual aids are used. On the board a detailed notation of the course of the calculation is made. Further, problems are analyzed under the teacher's guidance, and it is useful for consolidation to ask the children to do the examples in the textbook independently and write the solution of one of the problems analyzed.

It is important to note that, in lessons devoted to the consideration of general methods of addition within the bounds of 20 without carrying (and then with carrying) over ten, it is absolutely necessary to have the pupils work with individual didactic material. Here the children are able not only to reproduce all the actions of the teacher, which they have just seen, but also to perform the proper actions by analogy with the way in which they were done before. For example, after the cases of addition to nine and eight carrying over ten have been analyzed, the pupils may independently consider the cases of addition to seven, using for this purpose the same visual aids, and reasoning and acting by analogy.

For independent work it is useful to ask the children to examine the drawing and detailed outline of the solution of an example in the textbook, and to make in their notebooks the same kind of drawing and notation, but not, for example, for the case 7 + 5, analyzed in the textbook, but for the case 7 + 4 or 7 + 6 etc. (the assignment may be given in several variants).

To develop the children's powers of observation and their capacity for analysis and comparison, it is very important, beginning with the first grade, to require them frequently to establish the similarities and differences among various examples and problems. This purpose is met through exercises in composing examples by analogy with a given model. Examples of such an assignment are cited: Compose pairs of examples from the paradigm:

5 - 3   7 - 4   8 - 2   6 - 3
15 - 3   ____   ____   ____

or compose 3 examples from the paradigm:

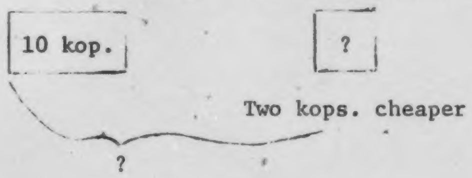
13 + 6 = 19   14 + 2 = 16
19 - 6 = 13   __________
19 - 13 = 6   __________

or compose examples like: 10 + 2, 10 + 3, 10 + 4 (usually the composition of examples is combined with their solution). Instruction in the solution of examples in addition and subtraction also continues to be used.

When instructing pupils to solve problems on increasing and decreasing numbers by several units, primary importance is attached to the children's correct understanding of the expression, "greater (smaller) by so many." For this reason, the first few times it is useful to return to independent sketching of the problem's conditions. The correctness of execution in such an assignment may serve as an indication that the pupils have understood the conditions and can go on to the independent solution of the problem.

If the teacher gives special consideration to instructing the children in the diagrammatic representation of the conditions and their "reading," this representation may very soon be used for independent work. The assignment will consist of composing a problem from a diagram and solving it. This work serves as the natural development of what was done before, in the consideration of problems on finding a sum or remainder. It is also good preparation for analogous work on compound problems containing simple problems of this type.

In the solution of compound problems, diagrams like the following are used for independent composition and solution.



It is useful to give such diagrams in pairs, so that they represent problems of both types: i.e., on increasing and on decreasing numbers by several units. Besides the composition and solution of problems, children may be asked to explain how and why their solutions differ.

Since, by the end of the second quarter, the children have more or less learned to read, it is imperative at this time to systematically conduct independent work dealing with the reading of the problem. After the children have read through the problem by themselves, the teacher checks their work with questions like "What [or who] is the problem talking about?" Then it is necessary to have the text read aloud or reproduced by the teacher. Later, the corresponding assignments become more complex. For example, the children may be asked to read its question aloud. To fulfill this requirement, the child must at least read through the whole text twice and mentally separate the question from the conditions. This is already the beginning of analysis, the most important factor in the solution of a problem. Another assignment may be given:

> Read through the problem and be ready to tell the significance of a certain datum, etc.

All these types of work are very important since the children are now really brought to a fully independent solution of simple problems.

During instruction in the solution of compound problems, many of them are solved under the teacher's guidance, but simple problems should usually be solved independently. Here it is advantageous that the teacher read the problem aloud. In all these cases the solution itself may be executed in diverse forms -- it may be "printed" by means of movable numerals, it may be written in full in the notebooks, or not written at all (only the answer being indicated). Finally, only the sign may be written, indicating what operation must be used to solve the problem. The use of these various forms of representation permits the children to solve more problems in the same time.

In the introduction to multiplication and division, primary significance is attached to disclosing the meaning of these operations and their relation to the operations of the first stage. Here, besides all the types of assignments for independent work on examples and problems which were used before and continue to be used in the study of this topic, it is useful to introduce assignments specially directed toward demonstrating these relations. From this point of view, exercises like replacing examples in addition with examples in multiplication, and vice versa, are useful.

In studying multiplication and division, exercises in composing all possible examples from three given numbers are especially useful (analogous to those in addition and subtraction). For example, 12 ÷ 3 = 4, 12 ÷ 4 = 3, 3 x 4 = 12, 4 x 3 = 12.

Assignments which require the pupil to fill in blanks in a series of numbers or to continue a series of numbers are also useful. For example, the children are asked to continue the series 2, 4, 6,... or 3, 6, 9... or fill in the blank in the series 1,3,... 7, 9, and other assignments.

It is often possible to organize reciprocal checking of the examples being solved so that the children sitting next to each other can exchange notebooks and check each other's solutions.

In solving problems in multiplication and division it is again useful to return to the illustration of conditions. A drawing (usually diagrammatic) helps to reveal the relationship between the quantities given in the problem and the unknown. Along with simple problems on multiplication and division, problems in two operations of the type considered earlier must continually be used as material for the children's independent work at this time, so that toward the end of the year the children can independently, consciously, and quickly solve any problem stipulated in the curriculum.

The topic "A Hundred" is only begun in the first grade. Only six hours of class time are set aside for it. In this time it is recommended that the children be introduced only to oral and written numeration within the bounds of 100. Assignments analogous to those described above, in work on oral and written numeration within the bounds of 20, can serve as material for independent work in these lessons.

The year comes to an end with a review of what has gone before. In the last two weeks of classes, the teacher has the opportunity to check once again the children's mastery of the material which was studied during the year, and fill in the gaps in the knowledge of each pupil in the class. The pupils' independent work in this concluding stage of instruction in the first grade must serve as a check, as well as helping to fill in the gaps. The first purpose is served best by written independent work, based on the material of one topic and representing, as fully as possible, its most important point. The second purpose is served by cards with individual assignments, containing the material which a given pupil has least mastered. Exercises reviewing what has gone before must have considerable diversity of content, as well as of form. The review of educational material must also be combined with perfecting the organizational side of independent work.

Pupils' Independent Work in the Second Grade

The arithmetic course in the second year of instruction is rich in diverse and crucial content. Here the children must acquire the fundamental methods of mental calculation with numbers within the bounds of 100, become acquainted with various uses for all four arithmetical operations, gain an understanding of the meaning of these operations in solving various practical problems, and finish studying the tables of multiplication and division.

The continuation of work directed toward developing, in the children, strong habits of conscious and quick computation with numbers less than 100 is the basis for the formation of computational skills in the next grades. How successfully this task is completed in the first and second grades determines, to a significant degree, the success of all further mathematical instruction.

In arithmetic instruction in the second grade, besides the development of computational skills, it is very important to form in the children a whole series of important arithmetical concepts. A feature of the concepts formed in the second grade is that, in many cases, the knowledge which the children must acquire in the arithmetic lessons comes into an apparent conflict with the knowledge which they acquired in previous experience. In second grade arithmetic lessons, the differention of like phenomena easily confused by the children is a particularly obvious problem. There is a need for special work directed toward the differentiation of two types of division (division into equal parts and division according to content), the two types of problems on addition and subtraction (expressed in direct and indirect forms), the two ways of comparing numbers (by subtraction and by division), the two ways of decreasing or increasing a number (by several units and several times), and others.

Taking this into consideration, the second grade curriculum stipulates that the children solve a large number of appropriate simple arithmetical problems which reveal the essence of the distinctions on which the differentiation of these concepts is based, using material which is concrete and close to the children's ideas and interests. Work on problems which are very diverse, not only in content but also in arithmetical essence, affords wide possibilities for further perfecting and deepening the knowledge, skills, and habits the children acquired in the first grade.

Along with the development of separate, individual skills necessary for independent problem solving (the ability to read the problem, illustrate its conditions, to pick out the data necessary in order to answer the question, to outline a plan for solution, etc.) in the second grade, the next step forward, in the simultaneous use of these individual skills in solving not only simple, but also compound problems, must be made. The curriculum stipulates the instruction of children in the second grade in the solution of problems in two or three operations, including all the types of simple problems with which they dealt in the first, and then in the second grades.

In determining the goals of instruction in the second grade, it is necessary not only to consider the curriculum for this grade, but also to think about the goal for which the teacher must prepare the children in the first two years of arithmetic instruction. With this approach it becomes clear that the most important task in the second grade (aside from those enumerated above) is to create conditions under which the children amass knowledge of a number of arithmetical facts, necessary for the generalizations stipulated by the third- and fourth-grade curricula. This requirement must be reflected both in work on problems and in work on examples.

Indeed, aside from the significance of the solution of examples in the development of computational skills, which was shown above (see the section devoted to the various types of exercises dealing with the solution of examples), work on examples affords broad possibilities for preparing the children to understand the relationships between separate arithmetical operations and among component operations, for acquainting the children with the composition of numbers from addends and factors, and with the laws of arithmetical operations. Account must be taken of all these conditions in the development of a system of children's independent work in arithmetic lessons in the second grade.

It follows from the goals formulated above, above all, that the basic content of children's independent work in the second grade must be the solution of arithmetical examples and problems (not only simple, but also compound) in order to develop the appropriate skills and habits. A place, moreover, must be set aside for exercises directed toward a deeper study of the features of the arithmetical material with which the children must deal. Below, we consider the concrete forms in which these requirements are realized in the study of the primary topics of the second-grade curriculum.

Pupils' independent work in lessons on the topic "The Four Operations within the Bounds of 20." The present topic is devoted to the review of what was studied in the first grade. Much attention must be given to reviewing the tables of addition and multiplication within the bounds of 20. It is also very important to freshen the children's memory of the devices and methods of computation with which they were acquainted earlier and the devices and methods dealing with problem solving. As always, the review must be organized so that it facilitates, to some degree, the enrichment of knowledge acquired earlier, and the perfection of the skills and habits just formed.

Pupils' independent work must occupy a relatively large place in the review lessons. Along with exercises of types well known to the children from the first year of instruction, it is useful to introduce several varieties so that, in executing the teacher's assignments, the children must look at the same material from another point of view. For this reason, aside from the usual training exercises dealing with the solution of prepared examples and problems, it is especially important to make use of assignments requiring a great deal of independent thought and initiative from the children. Thus, in reviewing addition and subtraction with and without carrying over ten, the assignments requiring the children to compose examples from a given model prove to be very useful. Models for these assignments are made so that the children, while executing the assignments, receive material for the composition of various instances of the operation. In our experiment, for example, the children were asked to compose two more pairs of examples from the model:

$6 + 3 =$ $\quad\quad$ $7 - 2 =$

$16 + 3 =$ $\quad\quad$ $17 - 2 =$

After solving the given examples and independently composing analogous ones, the children were asked to be prepared to explain the solutions they had reached. As a check, the teacher asked how the examples in each pair were alike and how they were different.

The children's independent construction of examples from a given answer is also frequently used as a review. For example, they were asked to compose any six examples with 18 for an answer. In this case, it depended on the pupils' own initiative whether they made up only examples which did not require carrying over ten, or whether they used numbers for which the ability to add and subtract carrying over ten was required. It also depended on the students' own initiative whether they used, let us say, only addition, or included subtraction as well, and finally, whether they composed examples on multiplication. This assignment can be given during the review of addition and subtraction without carrying over ten. However, by the way the children approach it, the teacher can tell approximately how well each of them remembers other instances of the operations from the first grade.

Further, because in future work in the study of addition and subtraction within the bounds of 100 the children's reasoning must often proceed analogously their reasoning in the study of addition and subtraction within the bounds of 20, we included, as early as the first weeks of the classes devoted to reviewing what had been covered, assignments which served as a certain preparation for such reasoning. The children were asked to compose examples analogous to the ones in the model, which used the first ten numbers, but using numbers beyond 10.

For example, as a model the teacher gave the children examples like: 3 + 5 = 8, 5 + 3 = 8, 8 - 5 = 3, 8 - 3 = 5. From this model they had to compose analogous examples with the numbers 18, 12, 6 and 20, 6, 14. Then the children were asked to independently compose any example in addition and then construct the corresponding subtraction example.

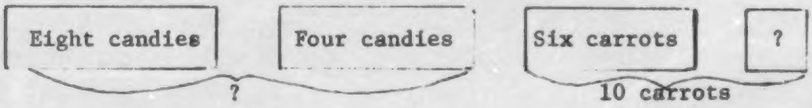
Such exercises represent a development of the work conducted in the first grade. They lay a wider foundation for the formation of the proper generalizations (about the link between addition and subtraction, and about the interdependence of the components of these operations); and they are good material for practice in drawing analogies. In drawing an analogy, in this case, the children must apply a regularity which was observed in smaller numbers to work with larger numbers. This kind of analogy is precisely what is necessary in preparing for the kind of reasoning which later must be relied upon when considering operations within the bounds of 100.

In the first lessons in the second grade, it is already quite possible--and very useful--to give the children practice in independently making comparisons by juxtaposing a pair of examples which differ by only one feature. In selecting examples for exercises, it is necessary, of course, to strive for the condition so that the conclusion which the children can reach through comparison acquires some cognitive meaning, i.e. deepens the knowledge which they have acquired earlier and serves as preparation for the following work. The following is a model that can be used to create the foregoing condition. Two examples are written on the board: 18 - 2 and 18 + 2. The teacher asks the children to solve them, to think about how they differ, and to explain why, in the solution of one example, the answer is greater than 18 and in the solution of the other it is less. In the check, the pupils explained that in these examples, the numbers are the same--18 and 2, but in the first it is necessary to take away 2 from 18 and in the second to add 2 to 18; that if 2 is taken away from 18, the number is smaller, and that if 2 is added, the number is larger than it was.

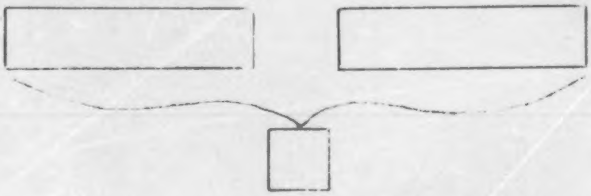
Not only do such exercises develop the children's powers of observation and capacity for the analysis and understanding of causal relations; they also help to deepen the knowledge of arithmetical operations which the children acquired in the first grade, where this deduction was not made in a generalized form.

Independent work on problems in the review of what was covered in the first grade must also be directed not only toward freshening the children's memory of what they learned in first grade, but also toward deepening this knowledge. Thus, in the first grade sufficient attention was given to the diagrammatic notation of the conditions of a problem on finding the sum of two numbers and on finding one of the addends from the sum and the other addend. The children, for example, knew how to make diagrams for problems of the following type:

> In one box there were eight pieces of candy, and in a second, four pieces. How many pieces of candy were there in all in the two boxes? There were 10 carrots in two bunches. In one bunch there were six. How many carrots are in the second bunch?



After reviewing with the children the notation for problems of this type and also the composition of a problem from a diagrammatic outline, the teacher may give the children a pair of these problems for independent work with the assignment to write both problems according to the following diagram:



The children must not only independently apply the familiar method of the diagrammatic representation of conditions, but must also unwittingly perceive the difference between the problems under consideration—a variation which requires the differential use of the same diagram and leads to different solutions.

In reviewing problems on increasing and decreasing a given number by several units, it is also very useful to construct assignments for the children's independent work so that, from the very beginning, the children compose and compare, in the course of doing them, corresponding pairs of concepts.

Exercises analogous in nature can also be carried out in a review of multiplication and division within the bounds of 20. Most of the pupils' attention must be directed toward reviewing the meaning of these operations. For this reason, both in solving problems and in reviewing the tables, it is useful to organize the children's independent work so as to deal with the illustration of a problem's conditions, and to reveal the meaning of an operation (replacement of multiplication by addition, and vice versa). It is best to organize the check of the mastery of the tables in the form of an "arithmetical dictation."

After the review of what was studied in the first grade, the children turn to the study of numeration and the four operations within the bounds of 100. We will consider the primary units of this topic.

Numeration and the four operations with whole numbers of tens. The children were acquainted with numeration within the bounds of 100 at the end of the first year of instruction, so this question must, on the whole, be considered as a review. What is new to the children in this topic are the operations with whole numbers of tens, and problems in two operations including multiplication and division.

The use of visual aids is very important in understanding operations with whole numbers of tens. Using counting sticks tied in "bundles" of 10 each, the teacher must make the children conscious of the fact that 10 sticks constitute 1 ten, and 1 ten is nothing other than 10 sticks (units). After the children gain an understanding of this principle through visual demonstration and through work under the teacher's direct guidance, all the operations with whole numbers of tens can be examined on the basis of the children's independent work. The children's independent work is the starting point in the lessons devoted to the study of each new instance. The independent work is built on the material of the first ten numbers in preparing for the study of the corresponding instances of operations with whole numbers of tens. It is also useful to make use, in assignments, of material which affords possibilities for the composition and comparison of corresponding instances of operations. For example: 3 + 5, 30 + 50, 8 - 6, 80 - 60.

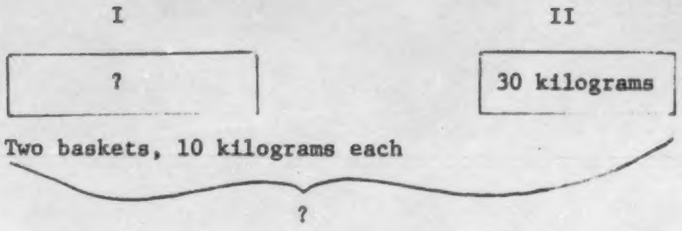
After the pupils consider, under the teacher's guidance, the illustrations and detailed notations given in the textbook to clarify new instances of operations with even tens, it is possible to ask them to try independently to gain an understanding of a notation relating to a new instance (for example, after they have already understood the addition illustrated on page 11 and the multiplication on page 15, division from a book can be used in conducting the children's independent work). The assignment may be given in this form:

> Carefully examine the solution to example 40 on page 17 of the textbook, show with the sticks all that is written there and be prepared to explain the solution of this example.

In exercises directed toward consolidating the acquired knowledge, it is important to include numerical material, not isolating operations with even tens, but combining work on them with other operations within the bounds of 20.

The possibility of using children's independent work when introducing problems of a new type, and of using their independent work on compound problems including multiplication or division was mentioned above. As preparation for solving such problems, one should review with the children all the methods and devices for work which they used in first grade for solving corresponding simple problems. Just before solving the new kind of problem, the children are asked to solve, in independent work, two problems analogous to those of which the new one is composed. After checking this work, the teacher can present the new problem, analyze its conditions with the children, explain that it is not possible to get the answer to the question at once, and then ask the children to solve it independently.

In some cases the diagrammatic notation, to which the children grew accustomed in the first grade, proves very useful. For example, in order to clarify to the children the method of solving problem 127 from the textbook, a diagrammatic representation (apart from the drawing in the book) is useful:



The diagram is made in the following way. The teacher reads the problem's text:

> Some schoolchildren gathered two baskets of apples with 10 kilograms in each basket from one apple tree, and, from another tree, 30 kilograms of apples. How many kilograms did the children gather from the two trees?

Then one of the children repeats the question and it is explained that they must find the kilograms of apples which were gathered from the two apple trees. Thus the diagram must have two boxes (as is done in the first grade in the solution of compound problems including the increasing or decreasing of a number by several units). The children are asked further, whether the number of kilograms of apples which were gathered from the first apple tree, and whether the number of kilograms of apples which were gathered from the second tree are stated in the problem. The appropriate data are written in the diagram (a question mark is put in box I, and "30 kilograms" is written in box II).

> What is stated in the problem about the first apple tree?

Again, the appropriate figure is written, but this time below the first box (as was done in the first grade, in the construction of diagrams for problems which require the increase or decrease of a number by several units). Finally, with the help of a bracket and question mark, it is indicated what must be found out in the problem.

After the conditions are analyzed and noted in the diagram, the children independently solve the problem. Later, in the solution of problems of the given type, one may begin to include in the children's independent work the diagrammatic representation of their conditions and the composition of problems from such representations. This work provides further development of the knowledge and skills acquired earlier, since the children learn to apply them under new conditions; this has great significance for instruction in problem solving.

Pupils' independent work in lessons on the topic "Additions and Subtractions within the Bounds of 100." This major topic requires approximately six weeks of class time. It is divided into two subtopics--addition and subtraction with, and without, carrying over ten. The study of new instances of the arithmetical operations is here interwoven with the introduction of new types of problems (problems in which it is necessary to increase or decrease a number by several units, indirect problems on finding an unknown minuend or unknown addend from the sum and the other addend, problems on finding the third addend, on comparing numbers by subtraction).

There is no major difference in the organization of children's independent work in the study of addition and subtraction both without carrying, and with carrying over ten, since both are equally familiar to the children from the first grade where they were studied using numbers within the bounds of 20. Hence, we will consider questions relating to the study of new instances of addition and subtraction as a group, and separately analyze questions connected with instruction in solving new types of problems.

The system for the study of various cases of addition and subtraction is very clearly defined in the textbook, which provides for a gradual shift from easier cases to more complex ones. The selection of numerical material for children's independent work should follow this system. Pupil's independent work in the study of each new instance of addition or subtraction should appear during preparation for the perception of new material, during this material's introduction, and during consolidation. Preparation for considering each new instance will most frequently consist of solving appropriate examples, using what was learned before.

For example, in the lesson on the introduction of addition, without carrying over ten, within the bounds of 100 (e.g., 45 + 3), the children may be given, as preparatory work, examples on addition within the bounds of 10, and also corresponding examples on addition within the bounds of 20, such as 15 + 3, 17 + 2, etc. It is also very useful at this stage of instruction to continue using practical exercises with visual aids. Here the same materials with which the corresponding instances of operations were explained in the first grade (counting sticks and bundles of sticks) are used. This makes it possible to demonstrate visually the similarity of the new cases to those which the children encountered working with numbers within the bounds of 20.

By gradually increasing the proportion of the children's independent participation in the study of new cases by analogy with familiar ones, it is possible, finally, to bring the children to the independent examination of new material as described above. This is relevant to addition and subtraction carrying over ten. Here it is useful to use visual aids analogous to those used in the first grade. There the device "The Second Ten," a demonstration board consisting of two rows of boxes with ten in each was used; here we propose the device described by G. B. Polyak called "Calculation Table. The First Hundred" [17:146-47].

In examining problems of the type 30 + 26 or 87 - 30, it is necessary, as preparation, to solve not only examples on addition and subtraction within the bounds of 10, but also examples on addition and subtraction with even tens. Since all the material which must be used in preparing for the study of the new topic is well known to the children, the teacher must try to construct assignments so that the independent work is not monotonous, using for this purpose various types of assignments dealing not only with the solution, but also with the children's independent composition of examples, which we described above. This is also true of exercises for independent work directed toward the consolidation of new knowledge. Especially significant is the use of assignments which require the children to make comparisons, establish points of similarity and difference between observed examples, and reason by analogy. The appropriate work is a development of what was outlined for the first lessons devoted to reviewing material already covered. Thus, so that the children may establish more precisely the similarity between cases of a single type of addition, using numbers of different magnitudes, it is possible to give the following assignment for independent work. Columns of examples are written on the board (it is even better if the corresponding cards are prepared for the individual work of each pupil):

6 + 3	8 + 2	7 + 5
16 + 3	18 + 2	17 + 5
26 + 3	28 + 2	27 + 5
36 + 3	38 + 2	37 + 5
...	...	...

The children are asked to continue these columns, constructing examples of the same type.

In checking the students' work, we established how the examples in each column differ from each other, and how the differences in examples lead to differences in solution. Thus, a general rule for the solution of problems of the given type is formulated. In completing the assignment, the children must not only perform the appropriate calculations, but also make comparisons between the examples they have solved; note the general principle by which they are arranged; independently compose, on this basis, the next examples; consider all the examples in each column as a whole; and draw a general conclusion about the method of solving them.

It is also useful to give, for comparison, examples in which the differences concern the method of computation. Thus, one column of examples may represent addition without carrying over ten, and the second, with carrying. In comparing these columns, the children must notice this feature, and themselves compose examples relating to each aspect.

All the examples carried out with material on the first twenty numbers in order to provide a deeper familiarity with the composition of numbers and properties of arithmetical operations, must be repeated with material on large numbers which the children first encounter in the second grade. The corresponding assignments will also be built around the transfer of earlier-acquired knowledge to a broader range of numbers (with the help of analogy). Some examples of such assignments follow.

Earlier the children did exercises in which they were required to indicate the composition of a given number according to a model:

17 = 10 + 7   12 = 10 + _____
14 = 10 + 4   16 = ______

Now the analogous exercise must be performed with the first hundred numbers:

36 = 30 + 6 58 = ________
27 = 20 + 7 43 = ________

In the first grade the children solved the so-called examples with blanks of the type 6 + ___ = 8 and others. Here, they can be given analogous examples with larger numbers: 26 + ___ = 29, 28 + ___ = 30, etc. Until this time the children used the commutative property of sums only with numbers less than 20. Now they can be given the opportunity to check it for larger numbers. With this purpose, they can be asked to compose examples from the model: 23 + 7 = ___, 7 + 23 = ___.

Solution of examples in two operations, as well as in one operation, should be included in the children's independent work. It is also useful to assign examples with one of the components left out. For example: 14 - 2 + ___ = 15. Various examples of this type can be introduced through exercises in the completion of "magic squares," which are perceived by the children as a kind of game and excite great interest. They are very useful for developing the skill of mental computation.

This gradual increase in the complexity of assignments dealing with the solution and composition of examples facilitates not only the formation of the proper computational skills, but also the children's deeper mastery of the methods of operation, properties of numbers, and relationships among the components of operation.

In instruction in solving new types of problems, the nature and place of the children's independent work depends on the characteristics of each type of problem. Several of the problems introduced do not cause the children any particular difficulty, since their solution is based entirely on what the children already know and requires only the application of knowledge and skills acquired earlier under somewhat altered conditions. In these cases, independent work can be given to the children from the very beginning at the stage of introduction. This was shown above, for example, in problems in which the increase or decrease of a number by several units was encountered twice.

In other cases, the examination of a new type of problem can be conducted through recourse to the children's acquired experience with practical operations with objects. This is true of problems on comparing numbers by subtraction. Here independent work can also serve as a starting point in the introduction of new material, but it will differ in nature from the preceding case. There the goal of independent work was to freshen the children's awareness of a series of arithmetical facts which they learned in the first grade, i.e., the realization of the knowledge and skills of problem solving, knowledge and skills which must be used in solving a new type of problem.

In a lesson devoted to the comparison of numbers by subtraction, we are not dealing with earlier-acquired knowledge applied under new conditions. The children do not yet have the knowledge which would allow them to independently solve a problem of this type. Here we only suggest that, in their practical experience, the children more than once have had to solve the problem of the comparison of two objects, that the very statement of the question may be familiar to them and that thus, if we use their practical knowledge, it will be easier to bring them to an understanding of the arithmetical essence of the problem.

Independent work preparing the pupils to examine a new kind of problem must thus be of a practical nature. The children can be asked, for example, to compare practically the length of two strips of paper, two tapes, etc. By performing the approximate practical operations, the children soon can understand what precisely must be determined in this type of problem, and what arithmetical operation corresponds to the practical operations which they used in solving the problem.

Finally, the pupils encounter problems which the knowledge they acquired earlier does not help to solve; the knowledge may even hinder the mastery of new material. We have in mind the so-called "problems expressed in indirect form"—problems on finding the unknown minuend from the subtrahend and difference, or on finding an unknown addend from the sum and other addend. Problems of this type have more than once attracted the attention of methodologists and psychologists. Their interest is determined by precisely this feature--that the children's study of new material is, in this case, in direct contradiction to what they learned before this time. Thus, although during the whole first year the children always dealt with problems in which the expressions "made in all," "brought more," "bought more," etc., invariably implied the operation of addition, and in which the expressions "gave away," "ate up," "was left," etc., implied subtraction, it will now be necessary, when solving problems on finding an unknown minuend or addend containing these same expressions, to apply the operations in a way opposite to that which seems to suggest itself to the child under the influence of previous experience in solving direct problems.

Keeping in mind the difficulties such a reversal causes the children, the teacher must, in this case, very carefully compose and prepare an explanation accompanied by visual material. (The most expedient form of visual aids for explaining to the children the process of solving indirect problems is dramatization, which permits the illustration not only of the components of operation, but also of the operations themselves; such illustration is especially important for problems of this type.)

Pupils' independent work can be used here only at the stage of consolidation, after the children, under the teacher's guidance, have gained an understanding of the special features of the new problems. Practical experience and special studies indicate that even after the children have understood the characteristics of these problems they continue, for a very long time, to make errors, confusing indirect problems with the corresponding direct ones.

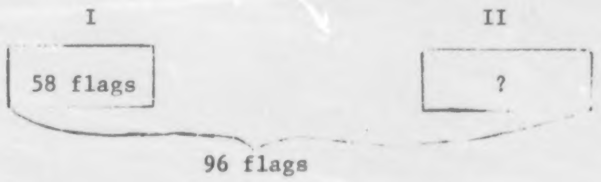
For this reason, when teaching children to solve indirect problems, it is very important to provide a selection of exercises for independent work which would afford sufficient material for discrimination and differentiation. For this purpose, it is useful, at this stage of instruction, to solve not only indirect problems, but the corresponding direct problems studied earlier as well. This excludes the possibility of solving problems mechanically, without sufficiently analyzing their conditions, or considering the specific characteristics of each type of problem.

However, one must do more than give direct and indirect problems alternately to the children for independent solution. It is also necessary to make sure that they have learned, when solving new indirect problems, to apply to the analysis of their conditions the devices and methods which they should have learned by this time, and also to see that they have mastered several new devices which prove especially useful in the solution of indirect problems.

In connection with this aspect of the solution of indirect problems, as work for the whole class and then as independent work, we successfully assigned diagrams and outlines of the conditions. An example is cited to illustrate how this work was conducted. The children were asked to independently solve the following problem, on finding one addend from the sum and other addend.

> To prepare for a holiday, the children made 58 flags in one day. The next day they made some more flags: there were 96 flags in all. How many flags did they make on the second day? (No. 336 from the second-grade textbook.)

The independent work was divided into two stages: (a) represent the problem's conditions by a diagram, and (b) solve the problem. The children were allowed to go on to the second stage of work only after the teacher had checked the diagrams of the conditions. The teacher conducted the check in the course of the work--walking up and down the aisles and looking over the pupils' notebooks. However, after making sure that all the children had been able to handle this part of the task, he submitted the task of checking the first stage of the work to general discussion. For this, one of the pupils was called on to write the problem's conditions on the board, explaining each step in his work. Other children on whom the teacher called participated in the explanation. The following diagram was written on the board:



The construction of the diagram was accompanied by an explanation.

> In one day the pupils made 58 flags--we will write that in box I. On the next day they made some more flags--since it is not stated how many, we must put a question mark in box II. Further, the problem says that there were 96 flags in all--that is how many they made in two days; we will draw a bracket and write down that in two days they made 96 flags. The problem asks how many flags did they make on the second day? We have a question mark in box II--we must answer this question.

After this analysis of conditions, the children solved the problem independently. It did not cause them any difficulty, since they recognized in the first diagram a problem of a type known to them since the first grade.

In the above case the diagram helped to indicate the general principle which unites problems on finding one addend from the sum and the other addend when they are expressed indirectly, and when the problem's formulation does not contain expressions which suggest the choice of one or another operation like ("In two bunches there are 20 radishes. In one there are 10. How many radishes are there in the other bunch?"). For this purpose the device of outlining the conditions was also used. In many cases, it facilitated the solution of indirect problems, since such a notation includes a whole series of separate expressions used in the complete text of the problem, and emphasizes the indirect nature of its formulation. An example is given as illustration.

> Asters were growing in a flower bed. The children picked six asters for a bouquet. After this, eight asters remained in the bed. How many asters were there in the bed at the beginning? (No. 266 from the textbook).

The outline of the conditions of this problem looks like this:

For the bouquet - 6 asters
Left in the bed - 8 asters
How many in all?

In writing the conditions of this problem one is half-way to solving it, since in this form it does not differ from problems well known to the children since first grade.

It is not very complicated to prepare the pupils for the independent execution of diagrams and outlines of the conditions of indirect problems if they have mastered these methods of representing various types of problems in preceding lessons. For consolidation and drill, it is continually necessary to include this type of assignment in independent work on new types of problems. Here again the knowledge, skills and habits which the children acquired in previous stages of instruction undergo development.

However, as was noted above, the solution of indirect problems is related to the use of still another way to approach the analysis of conditions, and the search for the method of solving a problem. We will deal with this in more detail.

The indirectness of formulation which hampers the understanding and solution of problems of this type is, in fact, still a formal indication; the problem's formulation may be changed so that the indirectness of formulation disappears, completly revealing the mathematical essence of the problem in the new formulation. An example illustrating this point follows.

> In a state farm, there were 16 tractors. When they sent some more tractors, there were 22 tractors in all on the farm. How many tractors did they send to the state farm? (No. 334 from the textbook).

In this formulation everything suggests addition to the pupil. Indeed--"There were, then they sent more...were in all.."--here not only are the separate expressions strongly associated in the children's minds, with the choice of this operation, but the course of the practical operation described in the conditions logically requires the performing of addition. As a result, even if the children correctly answer the question, they often write the problem's solution thus:

16 tr. + 6 tr. = 22 tr.

We will now formulate the same problem in another way.

> In a state farm, there are 22 tractors. Of these, 16 tractors were there earlier, and the rest were sent later. How many tractors were sent to the state farm?

We see that, from this rephrasing of the conditions, the essence of the problem does not suffer at all. Moreover the problem formulated in this way leaves no cause for doubt that it must be solved by subtraction (the children have encountered problems formulated in this way more than once even in the first grade).

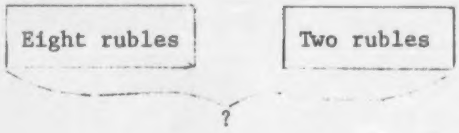
Accordingly, one of the devices facilitating the understanding and solution of indirect problems is this rephrasing of the conditions. An alteration in the formulation is one of the general devices which prove useful in the solution of other problems as well. In the work on the psychology of instruction which we have already quoted [3], this device is recommended as one of the distinctive means for facilitating problem solving. Thus, it is advisable, when the children are studying indirect problems, to acquaint them with this device, and to teach them to use it with awareness.

After carrying out the appropriate work with the teacher's guidance and help, the children may be assigned to change the formulation of a problem in their independent work on the conditions of this problem. A model of an appropriate assigment is cited: "Carefully read the problem and try to express it so that it is immediately clear how it is solved." The children must be given sufficient time to execute this assignment. Afterwards one should call on at least three or four pupils. The rest of the children should listen carefully to how they formulated the problem's text, and make suggestions for the correction and increased precision of the formulation. This task is the next step forward in instructing the children in the conscious reading of the conditions and their precise representation. The ability to express the same idea, the same relationships in a different form is one of the important indications of the pupils' development; hence such exercises have great educational significance.

In later exercises directed toward the consolidation of knowledge, skills, and habits acquired earlier, as we have said above, it is useful to include not only indirect problems, but also those directly-expressed with which the children confuse them. Here it is wise to formulate an assignment for the children's independent work which specially directs the pupils' thoughts toward the juxtaposition and comparison of these problems. So that this comparison may thoroughly reveal the peculiarities of these problems, one should vary the assignment, asking some of the children to diagram the conditions of both problems of the pair, allowing the children to establish the difference between them. In others, on the contrary, one should direct the work toward bringing out the similarities between indirect problems and the corresponding direct ones (as we showed above, with the example of the assignment of diagrams and outlines of the conditions). Finally, one should assign work in which the children must mark the points of similarity in comparing the formulation of indirect and direct problems, and underline the differences in the course of solution.

Along with these assignments it is constantly necessary to continue the work begun in the first grade, whose purpose is to develop in the children the ability to supply the question for data, to select the data necessary for answering a question, to compose a problem by analogy, etc. Here too in the exercises one may successfully follow the same principle on which the work on indirect problems was based. For example, it is possible to ask the children to compose two problems, one indirect, the other direct, from one diagram.

They are given the diagram:



The children are asked to compose one problem in whose conditions the words "were left" are used and another in whose conditions is the word "more." While checking the problems the children have composed, the teacher may ask them to solve both problems in the same way.

All during the work on the topic "Addition and Subtraction within the Bounds of 100," the children's independent work must consist of both the completely independent solution of simple problems of a type studied earlier, and the solution of compound problems which they solved in the first grade (using all the diverse forms of assignment used in the first grade).

Pupils' independent work in lessons on the topic "Tables of Multiplication and Division." This topic includes the study of all instances of multiplication and division by tables within the bounds of 100, and the introduction of various applications of these operations. Along with the construction of tables, their study, and practice exercises having as a goal the firm mastery of the tables of multiplication and division, much attention is devoted in this topic, as in the preceding one, to problem solving. Here the children first encounter problems on division according to content, on finding the parts of a number, and on increasing (and decreasing) a number by several times. They also encounter multiple comparison of numbers and problems solved by the method of reduction into units.

It is possible to regard the work carried out in the first grade as preparation for the study of multiplication and division within the bounds of 100. For this reason, here, as in the study of addition and subtraction, it often proves possible to prepare for and sometimes even to carry out the consideration of new material on the basis of the children's independent work.

Thus, as preparation for drawing up each new table, the children may be given diverse exercises on familiar material directed toward the review of the meaning of multiplication. For example: Problems requiring the replacement of addition with subtraction and vice versa, the continuation of an appropriate series of numbers (3, 6, 9, 12..., 4, 8, 12, 16 ...) to 100, and others.

During preparation for the construction of multiplication tables within the bounds of 100, the children can be asked to draw up independently the portion of the table which they learned in the first grade. For example, they can be asked to continue this table:

$3 + 3 = 6$	$3 \times 2 = 6$
$3 + 3 + 3 = 9$	$3 \times 3 = 9$
$3 + 3 + 3 + 3 = 12$	$3 \times 4 = 12$
.....	.....

It is not worthwhile to set any limits in the assignment--experience shows that many pupils construct the whole table of multiplication by three themselves, and not just within the bounds of 20. If there turn out to be many such children in the class, the teacher may let one of them put the new portion of the table on the board, including the other pupils in this work as well. In any case, after the construction of the first two or three tables, the rest may be made on the basis of the children's independent work. The teacher need only check on whether all the children have really understood how these tables are constructed, and organize further exercises directed toward their mastery.

When new tables are introduced, the children become acquainted with several new devices for selecting various addends. To make sure that they master these devices based on the properties of multiplication, it is necessary to include appropriate assignments in the pupils' independent work.

For example, the teacher may ask the children to write one of the examples in the multiplication tables directly. Let us say the example 4 x 8 was given. This example can be written in another way, as follows: 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4, 4 x 4 x 2, 4 x 2 x 2 x 2, etc. A detailed notation of the calculation can be used for this same purpose:

4 x 8 =
4 x 4 = 16
4 x 4 = 16
16 + 16 = 32

It is possible to give this representation as a model and ask the children to write other examples from this model (7 x 6, 2 x 8 etc.).

To consolidate knowledge of a table, it is possible to use all the exercises of the same type that were used in the study of addition and subtraction--the construction of examples from a given operation and one of the components (construct four examples on multiplication of six), the construction of examples from a given number, the solution of examples with a blank, and others. To establish the connection between multiplication and division, and also to introduce the commutative property of multiplication, assignments requiring the construction of examples from three given numbers (for example, 6, 4, 24), and all other types of tasks mentioned above, are useful.

It is useful to conduct the check on the children's mastery of the tables in the form of an arithmetical dictation. Here, however, it is already possible to include the children themselves in the check, organizing classwork in pairs so pupils sitting next to each other check each other's work, and in case of doubt, check with the table or ask the teacher.

In studying multiplication and division by tables, it is very important to conduct numerous practice exercises requiring the solution of prepared examples. The children must in the end learn the tables by heart. For this reason, it is useful to drill them more than once in the reproduction of the tables' results.

To increase the number of examples solved, it is useful to make frequent use of the so-called half-written tasks, in which the children write down only the answers to the examples they solve, without rewriting the conditions in their notebooks. This form of work may be used when solving problems from the textbook, as well as work from individual cards and from variants written on the board.

Now let us go on to consider questions relating to instruction in solving problems when studying a given topic, The content and nature of the pupils' independent work on problems, in this case, are determined, to a significant degree, by the features of the problems under consideration. Here, as during the study of addition and subtraction, the primary goal of problem solving is the formation of important arithmetical concepts. In the process of forming these concepts, the differentiation of similar concepts and operations must be ensured.

This is also relevant to problems on division according to content, which acquaint the children with the application of familiar operations under new conditions--i.e., solving a practical problem which is different in principle from earlier ones. The solution of these problems causes a series of difficulties connected with precisely the necessity of distinguishing this application of division from division into different parts, which the children have been studying until this time. The distinction here is one of principle, but it also involves the form in which they are written.

The difficulties connected with the necessity of distinguishing similar concepts arise also in the consideration of problems on increasing and decreasing numbers by several times, and in the comparison of numbers through division.

The children often confuse increasing (decreasing) a number by several times with the familiar instance of increasing (decreasing) a number by several units; decreasing gets confused with increasing. The children sometimes multiply when they try to solve problems on comparison through division, just because in the question there is the word "bigger" ("How many times bigger?"); comparison by division also gets confused with comparison by subtraction.

All this requires the wide use of juxtaposition and opposition of similar concepts during independent exercises on the material of these problems. The juxtaposition and comparison of various types of problems can here be carried out in the most diverse concrete forms.

Here, as in the cases described above (relative to problems on addition and subtraction), the work sometimes aims at the clarification of the similarities, and sometimes especially at the clarification of the differences between the problems.

We will not cite here supplementary examples of this work--they may easily be composed by the teacher, analogous to those described above. We note only that they must lead to the further development of the knowledge, skills and habits which were formed by the material of earlier problems.

For example, while the conditions of problems requiring increasing (or decreasing) a number by several units were formerly written diagrammatically and the illustration was given through full use of visual aids with objects (the children had to draw the number of objects indicated by the conditions), now these forms are gradually replaced by a diagrammatic illustration in the form of strips or line segments, drawn at least approximately to scale.

Thus, illustration takes on a conditioned nature. While earlier it was directed toward helping the children develop a concrete, graphic idea of the conditions, this new type of graphic illustration reflects in a visual form the relationships among the quantities given in the problem. This is the next serious advance in the development of school children's visual concrete thought processes.

At first the teacher himself makes such drawings of the conditions of a problem analyzed in class, directs the children's attention to the method of their execution and requires them to reproduce the problem's conditions from this drawing. Later he increasingly includes in the children's independent work the formulation of problems from a drawing, and the construction of a drawing to represent the conditions of a given problem.

The formulation of a question for data, and the selection of data necessary to answer this question, are included in the assignments for independent work, as they were before. This work must also become gradually more complex. We cite a concrete example. The children are given the conditions and numerical data:

> On one day a store sold eight boxes of apples; on the second day it sold four.

The assignment is formulated thus:

> Formulate a question such that the problem is solved by addition; then change the question so that it is solved by division.

At this stage of instruction, it is necessary to assign the children increasingly more often, the task of independently constructing problems of a definite type. These assignments will be formulated as follows:

> Compose a problem on increasing a given number by several times; or compose a problem for whose solution it is necessary to use division according to content, etc.

In the opposite assignment, when it is necessary to select the proper numerical data for a given question, it is very important to use material from the children's own observations--everyday numerical data which they have encountered in solving the preceding problems from the textbook, numerical data drawn from class excursions, etc. If this material from life, which may be used as a basis for the construction of problems, is systematically accumulated, if these numbers are fixed, written in special notebooks, used for making posters, etc., all this material will help in organizing the children's independent work in class and will allow the teacher to vary this work, making the assignments simpler or more complex at his discretion.

Thus, the teacher can, for example, introduce a poster on which various postal rates are written; the children are asked to compose problems in which it is necessary to calculate how much more expensive a stamped envelope is than an unstamped one, or how much more expensive various types of telegrams are, etc. This assignment will be relatively easy for the children, since they can draw the necessary data directly from a consideration of the poster. Somewhat more complicated is this assignment:

> Using this poster, compose a problem on the comparison of numbers by division in which the precise numerical data to be used are not indicated.

This type of assignment becomes more complicated, if the teacher gives the children freedom to choose any subject, or any data from those in their notebooks.

The work described above involving the children's independent construction of problems will strengthen the link between arithmetic instruction and life. Aside from simple problems directed toward the formation of the concepts repeatedly mentioned above, the children's independent work must also include the solution of compound problems. These must be both problems of new types, and those which were solved before.

* * * *

Since we limited our consideration to the fundamental topics of the curricula for the first and second years of instruction, we naturally could not completely describe all the aspects of assignments for independent work, or all the methodological devices and forms of organization used in carrying out these tasks during arithmetic lessons.

We set ourselves the goal of merely giving examples to illustrate those topics which, during the course of the work, answer the requirements and goals, advanced in preceding chapters, for organizing children's independent work.

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